

Matrices and Determinants

Miscellaneous Exercise Question Bank

Level – 1

1. (i) Use, $C_1 \rightarrow C_1 + C_2 + C_3 \Rightarrow C_1 = 0 \Rightarrow$ Determinant value = 0
- (ii) Use, $R_2 \rightarrow R_2 + R_3$ and then take $(a + b + c)$ out of the determinant which leads to R_1 equal to $R_2 \Rightarrow$ Determinant value = 0
- (iii) Use, $C_2 \rightarrow C_2 + C_3$ and then take $(ab + bc + ca)$ out of the determinant which leads to C_1 equal to $C_2 \Rightarrow$ Determinant value = 0
- (iv) Use, $C_1 \rightarrow C_1 + C_2 + C_3 \Rightarrow C_1 = 0 \Rightarrow$ Determinant value = 0
- (v) Use, $R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$

$$\Rightarrow D = \begin{vmatrix} 0 & (b-a)c & c(b-a)(a+b+c) \\ 0 & (c-b)a & a(c-b)(a+b+c) \\ 1 & ab & ab(a+b) \end{vmatrix} = ac(b-a)(c-a) \begin{vmatrix} 0 & 1 & a+b+c \\ 0 & 1 & a+b+c \\ 1 & ab & ab(a+b) \end{vmatrix} = 0$$
- (vi) Use, $R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$

$$\Rightarrow D = \begin{vmatrix} 0 & b(a-c) & \frac{(c-a)}{ac} \\ 0 & c(b-a) & \frac{(a-b)}{ab} \\ 1 & ca & \frac{1}{c} + \frac{1}{a} \end{vmatrix} = 0$$
- (vii) Use $R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$

$$\Rightarrow D = \begin{vmatrix} 0 & (a-b) & (a-b)(a+b+c) \\ 0 & (b-c) & (b-c)(a+b+c) \\ 1 & c & c^2 - ab \end{vmatrix} = (a-b)(b-c) \begin{vmatrix} 0 & 1 & (a+b+c) \\ 0 & 1 & (a+b+c) \\ 1 & c & c^2 - ab \end{vmatrix} = 0 \quad [\text{As } R_1 = R_2]$$
- (viii) $\begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} = abc - abc = 0$
2. (i) Use, $C_1 \rightarrow C_1 - C_2$ and $C_2 \rightarrow C_2 - C_3$

$$\Rightarrow D = (a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ -c & -a & ab \end{vmatrix} = (a-b)(b-c)(c-a). \text{ Hence proved}$$
- (ii) Use, $C_1 \rightarrow C_1 - C_2$ and $C_2 \rightarrow C_2 - C_3$

$$\Rightarrow D = \begin{vmatrix} a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \\ b-a & c-b & a+b \end{vmatrix} = (a-b)(b-c) \begin{vmatrix} 1 & 1 & c \\ a+b & b+c & c^2 \\ -1 & -1 & a+b \end{vmatrix}$$

Operate $R_1 \rightarrow R_1 + R_3$

$$= (a-b)(b-c) \begin{vmatrix} 0 & 0 & (c+a+b) \\ a+b & b+c & c^2 \\ -1 & -1 & a+b \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)(a+b+c). \text{ Hence proved}$$

(iii) Use, $C_1 \rightarrow C_1 - C_2$ and $C_2 \rightarrow C_2 - C_3$

$$\Rightarrow D = \begin{vmatrix} 0 & 0 & 1 \\ (a-b) & (b-c) & c \\ (a^3-b^3) & (b^3-c^3) & c^3 \end{vmatrix} = (a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ (a^2+ab+b^2) & (b^2+c^2+bc) & c^3 \end{vmatrix}$$

$$= (a-b)(b-c)(b^2+c^2+bc-a^2-ab-b^2)$$

$$= (a-b)(b-c)(c-a)(a+b+c). \text{ Hence proved}$$

(iv) Use, $C_1 \rightarrow C_1 - C_2$ and $C_2 \rightarrow C_2 - C_3$

$$\Rightarrow D = \begin{vmatrix} 0 & 0 & 1 \\ (a^2-b^2) & (b^2-c^2) & c^2 \\ (a^3-b^3) & (b^3-c^3) & c^3 \end{vmatrix} = (a^2-b^2)(b^3-c^3) - (a^3-b^3)(b^2-c^2)$$

$$= (a-b)(b-c)(c-a)(ab+bc+ca) \text{ Hence proved}$$

(v) Use, $C_1 \rightarrow C_1 - C_2$ and $C_2 \rightarrow C_2 - C_3$

$$\Rightarrow D = \begin{vmatrix} 0 & 0 & 1 \\ (a-b) & (b-c) & c \\ (a^4-b^4) & (b^4-c^4) & c^4 \end{vmatrix} = (a-b)(b^4-c^4) - (b-c)(a^4-b^4)$$

$$= (a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca). \text{ Hence proved}$$

3. (i) Use, $C_1 \rightarrow C_1 - C_3$ and $C_2 \rightarrow C_2 - C_3$

$$\Rightarrow D = \begin{vmatrix} (y+z-x) & 0 & x \\ 0 & (z+x-y) & y \\ (z-x-y) & (z-x-y) & (x+y) \end{vmatrix}$$

$$R_3 \rightarrow R_3 - R_1 - R_2$$

$$\Rightarrow D = \begin{vmatrix} (y+z-x) & 0 & x \\ 0 & (z+x-y) & y \\ -2y & -2x & 0 \end{vmatrix} = 4xyz. \text{ Hence proved}$$

(ii) Use, $R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$

$$\Rightarrow D = \begin{vmatrix} b^2 & -a^2 & b^2-a^2 \\ c^2-b^2 & c^2 & -b^2 \\ b^2 & a^2 & a^2+b^2 \end{vmatrix}$$

Use, $R_1 \rightarrow R_1 + R_3$

$$= \begin{vmatrix} 2b^2 & 0 & 2b^2 \\ c^2 - b^2 & c^2 & -b^2 \\ b^2 & a^2 & a^2 + b^2 \end{vmatrix}$$

Use, $C_1 \rightarrow C_1 - C_3$

$$= \begin{vmatrix} 0 & 0 & 2b^2 \\ c^2 & c^2 & -b^2 \\ -a^2 & a^2 & a^2 + b^2 \end{vmatrix} = 4a^2 b^2 c^2. \text{ Hence Proved}$$

(iii) Same as Q. 3(i) replace $x = a^2, y = b^2, z = c^2$

(iv) Use, $R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$

$$\Rightarrow D = \begin{vmatrix} -ab & bc - b^2 & c^2 \\ a^2 & -bc & ac - c^2 \\ ab & bc + b^2 & c^2 \end{vmatrix}$$

Use, $R_3 \rightarrow R_3 + R_1$

$$= \begin{vmatrix} -ab & bc - b^2 & c^2 \\ a^2 & -bc & ac - c^2 \\ 0 & 2bc & 2c^2 \end{vmatrix}$$

$$\begin{aligned} \text{Use, } R_1 \rightarrow R_1 + \frac{bR_2}{a} ; D &= \begin{vmatrix} 0 & bc - b^2 - \frac{b^2c}{a} & c^2 + bc - \frac{bc^2}{a} \\ a^2 & -bc & ac - c^2 \\ 0 & 2bc & 2c^2 \end{vmatrix} \\ &= -a^2 \left[2c^2 \left(bc - b^2 - \frac{b^2c}{a} \right) - 2bc \left(c^2 + bc - \frac{bc^2}{a} \right) \right] = 4a^2 b^2 c^2. \text{ Hence proved} \end{aligned}$$

4.(B) Given $4A^3 + 2A^2 + 7A + I = O$

Pre-multiply with A^{-1}

$$\Rightarrow A^{-1}[4A^3 + 2A^2 + 7A + I] = O \Rightarrow 4IA^2 + 2IA + 7I + A^{-1}I = O.A^{-1}$$

$$\Rightarrow I(4A^2 + 2A + 7) + A^{-1}I = O \Rightarrow A^{-1} = -(4A^2 + 2A + 7I).$$

5.(D) For the given matrix to be singular, we must have, $\begin{vmatrix} 3 & -1+x & 2 \\ 3 & -1 & x+2 \\ x+3 & -1 & 2 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 3 & -1+x & 2 \\ 0 & -x & x \\ x & -x & 0 \end{vmatrix} = 0, [R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1] \Rightarrow \begin{vmatrix} x+4 & -1+x & 2 \\ 0 & -x & x \\ 0 & -x & 0 \end{vmatrix} = 0, [C_1 \rightarrow C_1 + C_2 + C_3]$$

$$\Rightarrow (x+4)(0+x^2) = 0 \Rightarrow x = -4, 0$$

Note that only $-4 \in [-4, -1]$.

6.(B) Since $A^{-1} = \frac{\text{adj } A}{|A|}$ and if λ is eigen value of A , then λ^{-1} is eigen value of A^{-1} .

Thus for $\text{adj}(A)X = (A^{-1}X)|A| = |A|\lambda^{-1}I$.

Thus, eigen value corresponding to $\lambda = 3$ is $4/3$ and corresponding to $\lambda = -2$ is $4/-2 = -2$.

7.(A) $1 + P + P^2 + \dots + P^n = O$

Pre-multiplying both sides by P^{-1}

$$\Rightarrow P^{-1} + I + IP + \dots + P^{n-1}I = O.P^{-1} \Rightarrow P^{-1} + I(1 + P + P^2 + \dots + P^{n-1}) = O$$

$$\Rightarrow P^{-1} = -(1 + P + P^2 + \dots + P^{n-1})I \Rightarrow P^{-1} = -P^n = P^n$$

8.(A) $P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}_{3 \times 3} \begin{bmatrix} -1 & -2 \\ -2 & 0 \\ 0 & -4 \end{bmatrix}_{3 \times 2} \begin{bmatrix} -4 & -5 & -6 \\ 0 & 0 & 1 \end{bmatrix}_{2 \times 3}$

$$P = \begin{bmatrix} -3 & -14 \\ -8 & -20 \\ -11 & -26 \end{bmatrix}_{3 \times 2} \begin{bmatrix} -4 & -5 & -6 \\ 0 & 0 & 1 \end{bmatrix}_{2 \times 3}$$

$$P = \begin{bmatrix} 12 & 15 & 4 \\ 32 & 40 & 28 \\ 44 & 55 & 40 \end{bmatrix}_{3 \times 3} \Rightarrow P_{22} = 40.$$

9.(A) $\Delta = 1(\omega^{3n} - 1) + \omega^n(\omega^{2n} - \omega^{2n}) + \omega^{2n}(\omega^n - \omega^{4n})$

$$\Delta = [(\omega^3)^n - 1] + 0 + \omega^{2n}[\omega^n - (\omega^3)^n \cdot \omega^n]$$

$$\Delta = 1 - 1 + 0 + \omega^{2n}[\omega^n - \omega^n] = 0.$$

10.(D) Given $A = \begin{vmatrix} \sin(\theta + \alpha) & \cos(\theta + \alpha) & 1 \\ \sin(\theta + \beta) & \cos(\theta + \beta) & 1 \\ \sin(\theta + \gamma) & \cos(\theta + \gamma) & 1 \end{vmatrix}$

Operate $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$

$$\therefore A = \{\cos(\theta + \gamma) - \cos(\theta + \alpha)\} \{\sin(\theta + \beta) - \sin(\theta + \alpha)\} - \{\cos(\theta + \beta) - \cos(\theta + \alpha)\} \{\sin(\theta + \gamma) - \sin(\theta + \alpha)\}$$

$$= \sin(\beta - \gamma) - \sin(\beta - \alpha) - \sin(\alpha - \gamma), \text{ which is independent of } \theta.$$

11.(C) Each term in $\Delta_1 \times \Delta_2$ is the sum of three terms. So each entry in C_1 or C_2 or C_3 in $\Delta_1 \times \Delta_2$ is the sum of three terms. Hence, $\Delta_1 \times \Delta_2$ can be written as the sum of $3 \times 3 \times 3 = 27$ determinants

12.(C) $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} a+b+c-x & c & b \\ a+b+c-x & b-x & a \\ a+b+c-x & a & c-x \end{vmatrix} = 0 \Rightarrow (x - \sum a) \begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix} = 0$$

$$\Rightarrow x = \sum a = 0 \quad (\text{By hypothesis})$$

or $1\{(b-x)(c-x) - a^2\} - c\{c-x-a\} + b\{a-b+x\} = 0$ by expanding the determinant.

$$\text{or } x^2 - (a^2 + b^2 + c^2) + (ab + bc + ca) = 0 \quad \text{or } x^2 - \left(\sum a^2\right) - \frac{1}{2}\left(\sum a^2\right) = 0 \quad \left\{ \because a+b+c=0 \Rightarrow (a+b+c)^2 = 0 \right.$$

$$\Rightarrow \left. \sum a^2 + 2\sum ab = 0 \Rightarrow \sum ab = -\frac{1}{2}\sum a^2 \right\} \text{ or } x = \pm \sqrt{\frac{3}{2}\sum a^2} \quad \therefore \text{The solution is } x = 0 \text{ or } \pm \sqrt{\frac{3}{2}\sum a^2}.$$

13.(A) The system will have a non-zero solution, if

$$\Delta \equiv \begin{vmatrix} a^3 & (a+1)^3 & (a+2)^3 \\ a & a+1 & a+2 \\ 1 & 1 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} a^3 & 3a^2+3a+1 & 3(a+1)^2+3(a+1)+1 \\ a^2 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

by $C_2 \rightarrow C_2 - C_1; C_3 \rightarrow C_3 - C_2$

$$\Rightarrow 3a^2 + 3a + 1 - \{3(a+1)^2 + 3(a+1) + 1\} \text{ (expanding along } R_3 \text{)}$$

$$\Rightarrow -6(a+1) = 0 \Rightarrow a = -1.$$

14.(D) Fundamental concept.

15.(B) Given, $\begin{vmatrix} 2+x & 3 & 4 \\ 1 & -1 & 2 \\ x & 1 & -5 \end{vmatrix} = 0$

$$\Rightarrow (2+x)(5-2) - 3(-5-2x) + 4(1+x) = 0 \Rightarrow 6 + 3x + 15 + 6x + 4 + 4x = 0 \Rightarrow 13x = -25 \Rightarrow x = -\frac{25}{13}.$$

16.(C) $A(z) = A\left(\frac{x+y}{1+xy}\right) = \left[\frac{1+xy}{(1-x)(1-y)}\right]$

$$\begin{bmatrix} 1 & -\left(\frac{x+y}{1+xy}\right) \\ -\left(\frac{x+y}{1+xy}\right) & 1 \end{bmatrix} \therefore A(x) \cdot A(y) = A(z).$$

17.(C) $|A| = -20 \quad \therefore a_{23} = \frac{\text{Cofactor of } 6}{-20} = \frac{-8}{20} = \frac{-2}{5}.$

18.(A) We have $F(\alpha)F(-\alpha) = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$$\therefore F(-\alpha) = [F(\alpha)]^{-1}.$$

19.(BC) $\det(-A) = (-1)^n \det(A)$

$$\det(A^{-1}) = \frac{1}{\det(A)} = 1 \Rightarrow \det(Adj A) = |A|^{n-1} = 1$$

Hence, $|\omega A| = \omega^n |A| = 1$ only when $n = 3k, k \in \mathbb{Z}$

20.(AD) Given, $(A+B)^2 = A^2 + B^2 + 2AB$

or $(A+B)(A+B) = A^2 + B^2 + 2AB$ or $A^2 + AB + BA + B^2 = A^2 + B^2 + 2AB$ or $BA = AB$

or $\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ or $\begin{bmatrix} a+2 & -a+1 \\ b-2 & -b-1 \end{bmatrix} = \begin{bmatrix} a-b & 1+1 \\ 2a+b & 2-1 \end{bmatrix}$

The corresponding elements of equal matrices are equal.

$$a+2 = a-b, -a+1 = 2 \Rightarrow a = -1$$

$$b-2 = 2a+b, -b-1 = 1 \Rightarrow b = -2 \Rightarrow a = -1, b = -2$$

21.(ABC) Given, $AB = A$, $BA = B$

$$\Rightarrow B \times AB = B \times A \quad \text{Or} \quad (BA)B = B \quad \text{Or} \quad B^2 = B$$

$$\text{Also, } A \times B \times A = AB$$

$$\Rightarrow (AB)A = A \quad \Rightarrow A^2 = A$$

$$\text{Now } (A^T)^2 = (A^T \times A^T) = (A \times A)^T = (A^2)^T = A^T$$

$$\text{Similarly, } (B^T)^2 = B^T$$

Hence, A^T and B^T are idempotent

22.(AC) A is an orthogonal matrix.

$$\therefore AA^T = I$$

$$\text{or } \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{or } \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\Rightarrow a+4+2b=0, 2a+2-2b=0 \text{ and } a^2+4+b^2=9$$

$$\Rightarrow a+2b+4=0, a-b+1=0 \text{ and } a^2+b^2=5$$

$$\Rightarrow a=-2, b=-1$$

23.(ACD) Given, $A^2 + A + 2I = 0$

$$\Rightarrow A^2 + A = -2I$$

$$\Rightarrow |A^2 + A| = |-2I|$$

$$\Rightarrow |A| |A + I| = (-2)^n$$

$$\Rightarrow |A| \neq 0$$

Therefore, A is non-singular; hence, its inverse exists. Also, multiplying the given equation both sides with A^{-1} , we get

$$A^{-1} = -\frac{1}{2}(A + I)$$

24.(AC) We have,

$$\begin{aligned} \begin{bmatrix} 1 & \tan \theta \\ -\tan \theta & 1 \end{bmatrix}^{-1} &= \frac{1}{1 + \tan^2 \theta} \begin{bmatrix} 1 & -\tan \theta \\ \tan \theta & 1 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix} &= \cos^2 \theta \begin{bmatrix} 1 & -\tan \theta \\ \tan \theta & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan \theta \\ \tan \theta & 1 \end{bmatrix} = \cos^2 \theta \begin{bmatrix} 1 - \tan^2 \theta & -2 \tan \theta \\ 2 \tan \theta & 1 - \tan^2 \theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \\ \therefore a &= \cos 2\theta, b = \sin 2\theta \end{aligned}$$

25.(ACD) B is an idempotent matrix

$$\therefore B^2 = B$$

$$\text{Now, } A^2 = (I - B)^2 = (I - B)(I - B) = I - IB - IB + B^2$$

$$= I - B - B + B^2 = I - 2B + B = I - B = A$$

Therefore, A is idempotent. Again,

$$AB = (I - B)B = IB - B^2 = B - B^2 = B^2 - B^2 = O$$

$$\text{Similarly, } BA = B(I - B) = BI - B^2 = B - B = O.$$

26.(AB) Let $I = k = 1$ (say). Then, $A_i A_k = A_k A_i = A_i A_1$

$$A_i A_k = A_1 A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

$$A_2 A_2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore A_i A_k + A_k A_i = I + I = 2I$$

If $i \neq k$ let $i = 3$ and $k = 2$, then

$$A_i A_k = A_1 A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{bmatrix}$$

$$A_2 A_1 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \Rightarrow A_1 A_2 + A_2 A_1 = O$$

27.(AC) $\sin A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$ and $\cos A = \begin{bmatrix} \sin \theta & \cos \theta \\ \cos \theta & \sin \theta \end{bmatrix}$

$$\therefore |\sin A| = \cos^2 \theta + \sin^2 \theta = 1.$$

Hence, $\sin A$ is invertible.

$$\begin{aligned} \text{Also, } (\sin A) \times (\sin A)^T &= \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & 0 \\ 0 & \cos^2 \theta + \sin^2 \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \end{aligned}$$

Hence, $\sin A$ is orthogonal. Also, $2 \sin A \cos A = 2 \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \sin \theta & \cos \theta \\ \cos \theta & \sin \theta \end{bmatrix}$

$$= 2 \begin{bmatrix} 2 \sin \theta \cos \theta & \cos^2 \theta + \sin^2 \theta \\ \cos^2 \theta - \sin^2 \theta & 0 \end{bmatrix} = 2 \begin{bmatrix} \sin 2\theta & 1 \\ \cos 2\theta & 0 \end{bmatrix} \neq \sin 2A$$

28.(AB) $A^{-1} = \frac{\text{adj}(A)}{|A|}$

$$\therefore (\text{adj } A)^{-1} = \frac{\text{adj}(\text{adj } A)}{|\text{adj } A|} = \frac{|A|^{n-2} A}{|A|^{n-1}} = \frac{A}{|A|}$$

Also $A(\text{adj } A) = |A| I$

or $A^{-1}(\text{adj } A^{-1}) = |A^{-1}| I$ or $A^{-1}(\text{adj } A^{-1}) = \frac{I}{|A|}$

or $A A^{-1}(\text{adj } A^{-1}) = \frac{A \cdot I}{|A|}$ or $I(\text{adj } A^{-1}) = \frac{A}{|A|}$

or $(\text{adj } A^{-1}) = \frac{A}{|A|} = (\text{adj } A)^{-1}$

29.(ABD) See theory.

30.(ABD) $A = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ \cos \beta & \sin \beta & 0 \\ \cos \gamma & \sin \gamma & 0 \end{bmatrix} \begin{bmatrix} \cos \alpha & \cos \beta & \cos \gamma \\ \sin \alpha & \sin \beta & \sin \gamma \\ 0 & 0 & 0 \end{bmatrix}$

Thus, A is symmetric and $|A| = 0$; hence, singular and not invertible. Also, $AA^T \neq I$

31.(ABC) All are properties of diagonal matrix.

32.(ABC) Given that A and B commute, we have

$$AB = BA \quad (\because A \text{ and } B \text{ are symmetric}) \quad \dots (1)$$

Also, $A^T = A, B^T = B \quad \dots (2)$

$$(A^{-1}B)^T = B^T(A^{-1})^T = BA^{-1} \quad (\because \text{if } A \text{ is symmetric, } A^{-1} \text{ is also symmetric})$$

Also from Eq. (1),

$$ABA^{-1} = B \quad \dots (3)$$

or $A^{-1}ABA^{-1} = A^{-1}B$ or $IBA^{-1} = A^{-1}B$ or $BA^{-1} = A^{-1}B$

Hence, from Eq. (2),

$$(A^{-1}B)^T = A^{-1}B$$

Thus, $A^{-1}B$ is symmetric. Similarly, AB^{-1} is also symmetric.

$$\begin{aligned} \text{Also, } BA = AB \text{ or } (BA)^{-1} &= (AB)^{-1} \text{ or } A^{-1}B^{-1} = B^{-1}A^{-1} \text{ or } (A^{-1}B^{-1})^T = (B^{-1}A^{-1})^T \\ &= (A^{-1})^T(B^{-1})^T = A^{-1}B^{-1} \end{aligned}$$

Hence, $A^{-1}B^{-1}$ is symmetric.

33.(BC) Since A is skew-symmetric, $A^T = -A$. we have,

$$A^2 + I = O \Rightarrow A^2 = -I \text{ or } AA = -1$$

$$\Rightarrow A(-A) = I \Rightarrow AA^T = I$$

Again, we know that

$$|A| = |A^T| \text{ and } |kA| = k^n |A|$$

Where n is the order of A . Now,

$$A^T = (-1)^n \times A \quad \text{or} \quad |A^T| = (1)^n |A| \quad \text{or} \quad [1 - (-1)^n] |A| = 0$$

Hence, either $|A| = 0$ or $1 - (-1)^n = 0$, i.e., n is even. But $A^2 = O - I = -I$

$$\Rightarrow |A|^2 = (-1)^n |I| = (-1)^n \neq 0$$

Hence, the only possibility is that A is of even order.

34. The given determinants is the product of the determinants
$$\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & 0 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & 0 \end{vmatrix} = 0$$

35. $D_c = \begin{vmatrix} x^3 - 1 & 0 & x - x^4 \\ 0 & x - x^4 & x^3 - 1 \\ x - x^4 & x^3 - 1 & 0 \end{vmatrix}$ is the determinant formed by the cofactors of determinant $D = \begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix}$

Hence, $D_c = D^2 = 3^2 = 9$.

36.(ABD)

(A) Total number of different matrices $= {}^9C_5 = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 126$

(B) $\begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \Rightarrow {}^3C_1$ such matrices

$\begin{pmatrix} 2 & & \\ & 1 & \\ & & 1 \end{pmatrix} \Rightarrow {}^3C_1$ such matrices

$\Rightarrow 3 + 3 \times 3 = 12$ symmetric matrices

(C) First 3 matrices in which all diagonal elements are 2 are symmetric

$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ is non singular

$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$ is non singular

$\Rightarrow$ out of 12 symmetric 6 non singular & 6 singular matrices $\Rightarrow$ probability $= \frac{1}{2}$

(D) max. trace is 6 $\Rightarrow$ number of matrices $= {}^6C_2 = 15$

Matrices and Determinants

Miscellaneous Exercise Question Bank

Level – 2

37. $B = \text{adj}(\text{adj}(A)) = |A| A = -20 \begin{pmatrix} 1 & -1 & 0 \\ 2 & -2 & 5 \\ 0 & 4 & 3 \end{pmatrix}$

$$|A| = -6 - 20 + (6) = -20$$

$$\text{Trace}(B) = -40$$

$$A) \rightarrow q)$$

$$\sum_{i=1}^3 \sum_{j=1}^3 b_{ij} = -20(1 - 1 + 0 + 2 - 2 + 5 + 0 + 4 + 3)$$

$$= -240$$

$$B) \rightarrow r)$$

$$\left| \frac{\text{Adj}(A) \cdot B}{|A|^2} \right| = \frac{1}{|A|^6} |\text{Adj}(A)| |B|$$

$$= \frac{1}{|A|^6} |A|^2 |A|^4 = 1$$

$$C) \rightarrow p)$$

$$\frac{|B|}{|\text{Adj}(A)|} = \frac{|A|^4}{|A|^2} = |A|^2 = 400$$

$$D) \rightarrow s)$$

38. (A) $\begin{vmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 1 & 8 & -9 \end{vmatrix} = 2(18 - 8) - (-9 - 1) - 3(8 + 2) = 20 + 10 - 30 = 0$

$$\text{adj}(A) \cdot B = \begin{pmatrix} 10 & -15 & -5 \\ 10 & -15 & -5 \\ 10 & -15 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ -4 \\ 14 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad A) \rightarrow q)$$

(B) $\begin{vmatrix} 1 & 5 & 2 \\ 3 & -1 & 1 \\ 1 & 21 & 7 \end{vmatrix} = -28 - 5(20) + 2(63 + 1) = 0$

$$\text{adj}(A) \cdot B = \begin{pmatrix} -28 & 7 & 7 \\ -20 & 5 & 5 \\ 64 & -16 & -16 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix} \neq 0 \quad B) \rightarrow s)$$

(C) $\begin{vmatrix} 2 & -3 & 1 \\ 3 & 1 & -2 \\ 4 & 2 & -5 \end{vmatrix} = 2(-5 + 4) + 3(-15 + 8) + (6 - 4)$

$$= -2 - 21 + 2 \neq 0$$

$$C) \rightarrow p)$$

(D) $4a^2 + b^2 + 16c^2 - 2ab + 4bc - 8ac = 0$

$$\Rightarrow (2a - b)^2 + (b - 4c)^2 + (4c - 2a)^2 = 0 \quad \Rightarrow \quad 2a = b = 4c \neq 0$$

$$\Rightarrow \quad 3 \text{ equations represents a plane } x + y + z = 0 \quad D) \rightarrow r)$$

39. Let $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = 0$

$ad = bc = p$ (say)

with no zero entry

p	Number of matrices possible
1	1
2	4
3	4
4	9
5	4
6	16
7	4
8	16
9	9
10	4
12	16
14	4
15	4
16	9
18	16
20	4
21	4
24	16
25	1
27	4
28	4
30	4
32	4
35	4
36	9
40	4
42	4
45	4
48	4
49	1
54	4
56	4
63	4
64	1
72	4
81	1
	209

With zero entry

$$\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} = 81 \text{ choices}, \quad \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 81 \text{ choices}, \quad \begin{vmatrix} a & 0 \\ 0 & 0 \end{vmatrix} = 9 \equiv 171 \text{ matrices}$$

$$209 \times 171 = 35739$$

(B) $\sum D(A) = 0 \forall A \in S$, in which there is no zero entry

$$\sum_{acd \neq 0} \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = 9(1+2+3+\dots+9)^2 = 9 \times 45^2$$

$$\sum_{abd \neq 0} \begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = 9(1+2+\dots+9)^2 = 9 \times 45^2$$

$$\sum_{abc \neq 0} \begin{vmatrix} a & b \\ c & 0 \end{vmatrix} = -9(1+\dots+9)^2 = -9 \times 45^2$$

$$\sum_{ad \neq 0} \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} = (1+2+\dots+9)^2$$

$$\sum_{A \in S} D(A) = 10(45)^2 = 20250$$

(C) $\sum_{A \in S} T(A) = 1000(1+2+\dots+9) + 900(0+1+2+\dots+9)$

$$= 1900 \times 45 = 85500$$

(D) $\sum_{A \in S} T'(A) = 900(0+1+2+\dots+9) + 900(0+1+\dots+9) = 1800 \times 45 = 81000$

$A \rightarrow r, B \rightarrow p, C \rightarrow q, D \rightarrow s$

40.
$$\begin{bmatrix} {}^nC_{k-1} & 0 \\ 0 & {}^nC_k \end{bmatrix} \cdot \begin{bmatrix} {}^nC_k & 0 \\ 0 & {}^nC_{k+1} \end{bmatrix} = \begin{bmatrix} {}^nC_{k-1} & {}^nC_k & 0 \\ 0 & {}^nC_k & {}^nC_{k+1} \end{bmatrix}$$

$$\sum_{k=1}^{n-1} \begin{bmatrix} {}^nC_{k-1} & {}^nC_k & 0 \\ 0 & {}^nC_k & {}^nC_{k+1} \end{bmatrix} = \begin{bmatrix} {}^nC_0 {}^nC_1 + {}^nC_1 {}^nC_2 + \dots + {}^nC_{n-2} {}^nC_{n-1} & 0 \\ 0 & {}^nC_1 {}^nC_2 + {}^nC_2 {}^nC_3 + \dots + {}^nC_{n-1} {}^nC_n \end{bmatrix}$$

$$= \begin{bmatrix} {}^{2n}C_{n-1} - n & 0 \\ 0 & {}^{2n}C_{n-1} - n \end{bmatrix} \Rightarrow \begin{bmatrix} {}^{2n}C_{n-1} & 0 \\ 0 & {}^{2n}C_{n-1} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$A \rightarrow r, B \rightarrow q, C \rightarrow p, D \rightarrow s$

41. $(AB)^r = A^r B^r$

$$(AB)^{r+1} = A^{r+1} B^{r+1} \Rightarrow A^r B^r \cdot AB = A^{r+1} B^{r+1} \Rightarrow B^r A = AB^r$$

$$(AB)^{r+2} = A^{r+2} B^{r+2} \Rightarrow A^{r+1} B^{r+1} AB = A^{r+2} B^{r+2}$$

$$\Rightarrow B^{r+1} A = AB^{r+1}$$

$$\Rightarrow B B^r A = B \cdot AB^r = AB^{r+1}$$

$$\Rightarrow BA = AB$$

$A \rightarrow q, B \rightarrow r, C \rightarrow p, D \rightarrow s$

42. A-s; B-r; C-q, r; D-p

(A) The given determinant is $\Delta = \begin{vmatrix} x+2 & x+3 & x+5 \\ x+4 & x+6 & x+9 \\ x+8 & x+11 & x+15 \end{vmatrix}$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_2$, we have

$$\begin{aligned} \Delta &= \begin{vmatrix} x+2 & x+3 & x+5 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix} \\ &= 2 \begin{vmatrix} x & x+1 \\ 2 & 4 \\ 1 & 1 \end{vmatrix} \quad [\text{Applying } R_1 \rightarrow R_1 - R_2 \text{ and } R_3 \rightarrow R_3 - R_2] \\ &= 2 \begin{vmatrix} x & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} \quad [\text{Applying } C_2 \rightarrow C_2 - C_1 \text{ and } C_3 \rightarrow C_3 - C_2] \\ &= -2 \quad [\text{Expanding along } R_3] \end{aligned}$$

(B) $\begin{vmatrix} 7 & 6 & x^2 - 13 \\ 2 & x^2 - 13 & 2 \\ x^2 - 13 & 3 & 7 \end{vmatrix}$

Let $x^2 - 13 = t$. Then

$$t^3 - 67t + 126 = 0 \Rightarrow t = -9, 2, 7 \Rightarrow x = \pm 2, \pm \sqrt{20}, \pm \sqrt{15}$$

Hence, sum of other five roots is 2.

(C) $\Delta = \begin{vmatrix} \sqrt{6} & 2i & 3 + \sqrt{6} \\ \sqrt{12} & \sqrt{3} + \sqrt{8}i & 3\sqrt{2} + \sqrt{6}i \\ \sqrt{18} & \sqrt{2} + \sqrt{12}i & \sqrt{27} + 2i \end{vmatrix}$

Taking $\sqrt{6}$ common from C_1 , we get

$$\begin{aligned} \Delta &= \sqrt{6} \begin{vmatrix} 1 & 2i & 3 + \sqrt{6} \\ \sqrt{2} & \sqrt{3} + 2\sqrt{2}i & 3\sqrt{2} + \sqrt{6}i \\ \sqrt{3} & \sqrt{2} + 2\sqrt{3}i & 3\sqrt{3} + 2i \end{vmatrix} \\ \text{Applying } R_2 &\rightarrow R_2 - \sqrt{2}R_1 \text{ and } R_3 \rightarrow R_3 - \sqrt{3}R_1, \text{ we get} \\ \Delta &= \sqrt{6} \begin{vmatrix} 1 & 2i & 3 + \sqrt{6} \\ 0 & \sqrt{3} & \sqrt{6}i - 2\sqrt{3} \\ 0 & \sqrt{2} & 2i - 3\sqrt{2} \end{vmatrix} = \sqrt{6} \begin{vmatrix} \sqrt{3} & \sqrt{6}i - 2\sqrt{3} \\ \sqrt{2} & 2i - 3\sqrt{2} \end{vmatrix} \\ &= \sqrt{6} \begin{vmatrix} \sqrt{3} & -2\sqrt{3} \\ \sqrt{2} & -3\sqrt{2} \end{vmatrix} \quad [\text{Applying } C_2 \rightarrow C_2 - \sqrt{2}i C_1] \\ &= \sqrt{6}(-3\sqrt{6} + 2\sqrt{6}) = -6, \text{ which is an integer} \end{aligned}$$

$$(D) \quad f(\theta) = \begin{vmatrix} \cos^2 \theta & \cos \theta \sin \theta & -\sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta & \cos \theta \\ \sin \theta & -\cos \theta & 0 \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + (\sin \theta)R_3$ and $R_2 \rightarrow R_2 - (\cos \theta)R_3$, we get

$$f(\theta) = \begin{vmatrix} 1 & 0 & -\sin \theta \\ 0 & 1 & \cos \theta \\ \sin \theta & -\cos \theta & 0 \end{vmatrix} = \sin^2 \theta + \cos^2 \theta = 1 \Rightarrow f(\pi/3) = 1$$

43. **A**-p, q, r; **B**-q; **C**-s; **D**-p, q, r.

(A) Multiplying C_1 by a , C_2 by b and C_3 by c , we obtain

$$\Delta = \frac{1}{abc} \begin{vmatrix} \frac{a}{c} & \frac{b}{c} & -\frac{a+b}{c} \\ -b+c & b & c \\ \frac{a}{b(b+c)} & \frac{a}{b(a+2b+c)} & \frac{a}{b(a+b)} \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get

$$\Delta = \frac{1}{abc} \begin{vmatrix} 0 & \frac{b}{c} & -\frac{a+b}{c} \\ 0 & \frac{b}{c} & \frac{c}{a} \\ 0 & \frac{b(a+2b+c)}{ac} & -\frac{b(a+b)}{ac} \end{vmatrix}$$

This shows that Δ is independent of a , b , and c .

(B) Applying $C_1 \rightarrow C_1 - (\cot b)C_2$, we get

$$\begin{aligned} \Delta &= \begin{vmatrix} 0 & \sin a \sin b & \cos a \\ 0 & \cos a \sin b & -\sin a \\ -\sin a / \sin b & \sin a \cos b & 0 \end{vmatrix} \\ &= -\frac{\sin a}{\sin b} [-\sin b \sin^2 a - \cos^2 a \sin b] \quad [\text{expanding along } C_1] \\ &= \sin a \end{aligned}$$

(C) Taking $1 / \sin a \cos b$, $1 / \sin a \sin b$, $1 / \cos a$ common from C_1, C_2, C_3 , respectively, we get

$$\Delta = \frac{1}{\sin^2 a \cos a \sin b \cos b} \Delta_1$$

where $\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ -\cot a & -\cot a & \tan a \\ \tan b & -\cot b & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 0 & -\cot a & \tan a \\ 1 / \sin b \cos b & -\cot b & 0 \end{vmatrix}$

Applying $C_1 \rightarrow C_1 - C_2$, we get

$$\Delta = \frac{1}{\sin b \cos b} [\tan a + \cot a] = \frac{1}{\sin a \cos a \sin b \cos b}$$

$$\begin{aligned}
 \text{(D)} \quad & \begin{vmatrix} a^2 & b \sin A & c \sin A \\ b \sin A & 1 & \cos A \\ c \sin A & \cos A & 1 \end{vmatrix} = \begin{vmatrix} a^2 & a \sin B & a \sin C \\ a \sin B & 1 & \cos A \\ a \sin C & \cos A & 1 \end{vmatrix} \\
 & = a^2 \begin{vmatrix} 1 & \sin B & \sin C \\ \sin B & 1 & \cos A \\ \sin C & \cos A & 1 \end{vmatrix} = a^2 \begin{vmatrix} 1 & 0 & 0 \\ \sin B & 1 - \sin^2 B & \cos A - \sin B \sin C \\ \sin C & \cos A - \sin B \sin C & 1 - \sin^2 C \end{vmatrix}
 \end{aligned}$$

[Applying $C_2 \rightarrow C_2 - (\sin B)C_1$ and $C_3 \rightarrow C_3 - (\sin C)C_1$]

$$\begin{aligned}
 & = a^2 [\cos^2 B \cos^2 C - (\cos A - \sin B \sin C)^2] \\
 & = a^2 [\cos^2 B \cos^2 C - (\cos(B+C) + \sin B \sin C)^2] \\
 & = a^2 [\cos^2 B \cos^2 C - \cos^2 B \cos^2 C] \\
 & = 0
 \end{aligned}$$

44. A-s; B-p; C-q; D-r

(A) Since A is idempotent, we have

$$\begin{aligned}
 & A^2 = A \\
 \Rightarrow & A^3 = AA^2 = AA = A^2 = A, A^4 = AA^3 = AA = A^2 = A \\
 \Rightarrow & A^n = A \\
 \Rightarrow & (I - A)^n = {}^nC_0 I - {}^nC_1 A + {}^nC_2 A^2 - {}^nC_3 A^3 + \dots \\
 & = I + (-{}^nC_1 + {}^nC_2 - {}^nC_3 + \dots)A = I + [({}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \dots) - {}^nC_0]A = I - A
 \end{aligned}$$

(B) A is involutory. Hence,

$$\begin{aligned}
 & A^2 = I \\
 \Rightarrow & A^3 = A^5 = \dots = A \text{ and } A^2 = A^4 = A^6 = \dots = I \\
 \Rightarrow & (I - A)^n = {}^nC_0 I - {}^nC_1 A + {}^nC_2 A^2 - {}^nC_3 A^3 + \dots \\
 & = {}^nC_0 I - {}^nC_1 A + {}^nC_2 I - {}^nC_3 A + {}^nC_4 I - \dots \\
 & = ({}^nC_0 + {}^nC_2 + {}^nC_4 + \dots)I - ({}^nC_1 A + {}^nC_3 A + {}^nC_5 A + \dots) \\
 & A = 2^{n-1}(I - A) \\
 \Rightarrow & [(I - A)^n]A^{-1} = 2^{n-1}(I - A)A^{-1} = 2^{n-1}(A^{-1} - I)
 \end{aligned}$$

(C) If A is nilpotent of index 2, then

$$\begin{aligned}
 & A^2 = A^3 = A^4 \dots = A^n = 0 \\
 \Rightarrow & (I - A)^n = {}^nC_0 I - {}^nC_1 A + {}^nC_2 A^2 - {}^nC_3 A^3 + \dots \\
 & = I - nA + 0 + 0 + \dots \\
 & = I - nA
 \end{aligned}$$

(D) A is orthogonal. Hence,

$$\begin{aligned}
 & AA^T = I \\
 \Rightarrow & (A^T)^{-1} = A
 \end{aligned}$$

45. A-r; B-s; C-p, r; D-p, q, r, s.

(A) Since A is idempotent, $A^2 = A^3 = A^4 = \dots = A$. Now,

$$(A + I)^n = I + {}^nC_1 A + {}^nC_2 A^2 + \dots + {}^nC_n A^n$$

$$= I + {}^nC_1 A + {}^nC_2 A + \dots + {}^nC_n A = I + ({}^nC_1 + {}^nC_2 + \dots + {}^nC_n)A = I + (2^n - 1)A$$

$$\Rightarrow 2^n - 1 = 127 \Rightarrow n = 7$$

(B) We have,

$$(I - A)(I + A + A^2 + \dots + A^7)$$

$$= I + A + A^2 + \dots + A^7 + (-A - A^2 - A^3 - A^4 - \dots - A^8)$$

$$= I - A^8 = I \text{ (if } A^8 = 0)$$

(C) Here matrix A is skew-symmetric and since $|A| = |A^T|$

$$= (-1)^n |A|$$
, so $|A|(1 - (-1)^n) = 0$. As n is odd, hence $|A| = 0$.
Hence, A is singular.

(D) If A is symmetric, A^{-1} is also symmetric for matrix of any order.

46. A-q; B-p; C-s; D-r

(A) $|A| = 2 \Rightarrow |2A^{-1}| = 2^3 / |A| = 4$

(B) $|adj(adj(2A))| = |2A|^4 = 2^{12} |A|^4 = 2^{12} / 2^{12} = 1$

(C) $(A + B)^2 = A^2 + B^2$

$$\Rightarrow AB + BA = 0 \Rightarrow |AB| = |-BA| = -|BA| = -|AB|$$

$$\Rightarrow |AB| = 0 \Rightarrow |B| = 0$$

(D) Product ABC is not defined.

47.(B) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\Rightarrow A^2 - (a + d)A + (ad - bc)I$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - (a + d) \begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} - \begin{bmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{bmatrix} + \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = O$$

48.(B) If $A = O$, $tr(A) = 0$, Suppose $A \neq O$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $|A| = 0$ and $A^2 - (a + d)A + (ad - bc)I = 0$

$$\Rightarrow a + d = 0$$

49.(D) $(I + A)^{100} = {}^{100}C_0 I^{100} + {}^{100}C_1 I^{99} A + \dots + {}^{100}C_{100} A^{100}$

$$= I + 100A + O + O + \dots + O$$

$$= I = 100A$$

50.(C) $AB = A \Rightarrow |AB| = |A|$

$$\Rightarrow |A| = 0 \text{ or } |B| = 1 \dots \dots \dots (1)$$

$$BA = B \Rightarrow |BA| = |B| \quad \dots\dots\dots(2)$$

$$\Rightarrow |A| = 1 \text{ or } |B| = 0$$

If $|A| = 0$, then from equation (2), $|B| = 0$

If $|B| = 0$, then from equation (1), $|A| = 0$

51.(C) $AB = A, BA = B$

$$ABA = A^2 \text{ or } A(BA) = A^2 \text{ or } AB = A^2 \text{ or } A = A^2$$

Similarly, $B^2 = B$

$$(A + B)^2 = A^2 + B^2 + AB + BA$$

$$= A + B + A + B = 2(A + B)$$

$$(A + B)^3 = (A + B)^2(A + B) = 2(A + B)^2 = 2^2(A + B)$$

$$\Rightarrow (A + B)^7 = 2^6(A + B) = 64(A + B)$$

52.(C) $(A + I)^5 = I + 5A + 10A^2 + 10A^3 + 5A^4 + A^5$

$$= I + 5A + 10A + 10A + 5A + A \quad (\because A^2 = A \Rightarrow A^3 = A^4 = A^5 = \dots = A)$$

$$= I + 31A$$

53.D & 54.C

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow B = \begin{bmatrix} a_{12} + a_{13} & a_{11} + a_{13} & a_{11} + a_{12} \\ a_{22} + a_{23} & a_{21} + a_{23} & a_{21} + a_{22} \\ a_{32} + a_{33} & a_{31} + a_{33} & a_{31} + a_{32} \end{bmatrix}$$

$$\Rightarrow X = A^{-1}B$$

$$\frac{1}{|A|} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} a_{12} + a_{13} & a_{11} + a_{13} & a_{11} + a_{12} \\ a_{22} + a_{23} & a_{21} + a_{23} & a_{21} + a_{22} \\ a_{32} + a_{33} & a_{31} + a_{33} & a_{31} + a_{32} \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} 0 & |A| & |A| \\ |A| & 0 & |A| \\ |A| & |A| & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow [A^{-1}B] = 2 \quad \text{Or} \quad |A^{-1}B| = 2 \quad \text{Or} \quad |B| = 2|A|$$

55.B, 56.D, 57.B

$$A^n - A^{n-2} = A^2 - I \Rightarrow A^{50} = A^{48} + A^2 - I$$

Further,

$$A^{48} = A^{46} + A^2 - I$$

$$A^{46} = A^{44} + A^2 - I$$

$$\vdots \quad \vdots \quad \vdots$$

$$A^{46} = A^{44} + A^2 - I$$

$$\text{Here, } A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^{50} = \begin{bmatrix} 25 & 0 & 0 \\ 25 & 25 & 0 \\ 25 & 0 & 25 \end{bmatrix} - 24 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

$$\therefore |A^{50}| = 1$$

Also, $\text{tr}(A^{50}) = 1+1+1 = 3$. Further,

$$\begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 25 \\ 25 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cup_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Similarly, } \cup_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \cup_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \cup = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ i.e., } |\cup| = 1$$

58.C $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix} \Rightarrow \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$$\Rightarrow A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 4 \\ -1 & -1 & -2-\lambda \end{bmatrix}$$

$$\Rightarrow \det(A - \lambda I) = -(\lambda - 1)(\lambda + 1)(\lambda - 3)$$

Thus, the characteristic roots are $-1, 1$ and 3 .

59.D Option (A) is not correct since its characteristic determinant is $\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$

The characteristic equation is $\lambda^2 - 1 = 0$. Therefore, $\lambda = 1, -1$

Hence, eigenvalues are 1 and -1 .

We similarly note that matrices given in options (B) and (C) have eigenvalues 1 and -1 . Hence, they are not correct.

Option (D) has characteristic equation $(1 - \lambda)^2 = 0$. Hence, eigenvalues are not 1 and -1 .

60.B Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$\Rightarrow A - \lambda I = \begin{bmatrix} a_1 - \lambda & b_1 & c_1 \\ a_2 & b_2 - \lambda & c_2 \\ a_3 & b_3 & c_3 - \lambda \end{bmatrix}$$

$$\Rightarrow \det(A - \lambda I) = (a_1 - \lambda)[(b_2 - \lambda)(c_3 - \lambda) - b_3 c_2] - b_1[a_2(c_3 - \lambda) - a_3 c_2] + c_1[a_2 b_3 - a_3(b_2 - \lambda)]$$

Now one of the eigenvalues is zero, so one root of equation should be zero. Therefore, constant term in the above polynomial is zero.

$$\therefore a_1 b_2 c_3 - a_1 b_3 c_2 - b_1 a_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - c_1 a_3 b_2 = 0 \quad (\text{collecting constant terms})$$

But this value is value of determinant of A .

$$\therefore \det A = 0$$

61.C As second row of all the options is same, we have to look at the elements of the first row. Let the left inverse be $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$.

$$\text{Then } \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore a + b + 2c = 1$$

$$-a + b + 3c = 1, \text{ i.e., } b = \frac{1-5c}{2}, a = \frac{1+c}{2}$$

Thus, matrices in the options (A), (B), and (D) are the inverses and matrix in option (C) is not the left inverse.

62.D Let right inverse be $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Then,

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Now, } a - c + 2e = 1$$

$$b - d + 2f = 0$$

$$2a - c + e = 0$$

$$2b - d + f = 1$$

This system of equations has infinite solutions.

63.C By observation there cannot be any left inverse for options (B) and (D). So we will check for (A) and (C) only.

$$\text{For option (A), let the left inverse be } \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}. \text{ Then, } \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ -3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, $a - 3b = 1$, $2a + 2b = 0$ and $4a + b = 0$ which is not possible. For option (C).

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & -3 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow a + 2b + 5c = 1, 4a - 3b + 4c = 0, d + 2e + 5f = 0, 4d - 3e + 4f = 1$$

Therefore, there are infinite number of left inverses.

$$\begin{bmatrix} 1 & 4 \\ 2 & -3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow a + 4d = 1$$

$$2a - 3d = 0$$

$$5a + 4d = 0 \text{ which is not possible}$$

Therefore, there is no right inverses.

64-66. $(f(-1)+8)x^2 + (f(2)-1)x + f(4)+3 = 0$

Has 3 roots a, b, c

$$\Rightarrow f(-1) = -8, f(2) = 1, f(4) = -3$$

Let $f(x) = ax^2 + bx + c$

$$\Rightarrow a - b + c = -8$$

$$4a + 2b + c = 1$$

$$16a + 4b + c = -3$$

$$\Rightarrow a = -1, b = 4, c = -3$$

$$\Rightarrow f(x) = -x^2 + 4x - 3$$

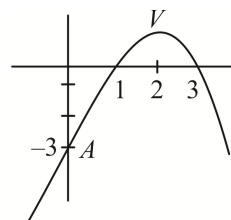
$$V(2,1), A(0,-3), B(x_3, y_3)$$

$$\frac{y_3 - 1}{x_3 - 2} = \frac{-1}{2}$$

$$2(-x_3^2 + 4x_3 - 3 - 1) = -x_3 + 2$$

$$2x_3^2 - 9x_3 + 10 = 0$$

$$x_3 = \frac{5}{2}, y_3 = \frac{3}{4}$$



64.(A) $\begin{vmatrix} -8 & -3 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 8 = 4 \times \text{slope of AV}$

65.(A) $XY = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} = \begin{pmatrix} x_1y_1 & x_1y_2 & x_1y_3 \\ x_2y_1 & x_2y_2 & x_2y_3 \\ x_3y_1 & x_3y_2 & x_3y_3 \end{pmatrix}$

$$\text{Trace} = 0 + 2 + \frac{15}{8} = \frac{31}{8}$$

66.(C) $XY = \begin{pmatrix} 0 & 0 & 0 \\ -6 & 2 & \frac{3}{2} \\ \frac{5}{2} & -\frac{15}{2} & \frac{15}{8} \end{pmatrix}$

$$XY \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \text{ represents, 2 planes, hence a line.}$$

67.(B) $\begin{bmatrix} a & c \\ c & b \end{bmatrix}$ can be made in $7 \cdot 7 \cdot 7 = 7^3$ ways of which $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is symmetric as well as skew symmetric.

68.(A) $\begin{vmatrix} a & c \\ c & b \end{vmatrix} = ab - c^2$ to be divisible by 7

If $a=0, b \neq 0 \Rightarrow c=0 \Rightarrow 6$ matrices or $b=0, a \neq 0 \Rightarrow c=0 \Rightarrow 6$ matrices

$a=0=b \Rightarrow 1$ matrix

Let $a \neq 0, b \neq 0$ then $c \neq 0$ ie $a, b, c \in \{1, 2, 3, 4, 5, 6\}$

If $a=b=c \Rightarrow 6$ matrices

If $a=b \neq c$ then $a+c$ must be divisible by 7 ie.

$\Rightarrow 6$ matrices

Let $a \neq b \neq c$

a	b	c	Number of matrices
1	2	3, 4	4
1	3		0
1	4	2, 5	4
1	5		0
1	6		0
2	3		0
2	4	1, 6	4
2	5		0
2	6		0
3	4		0
3	5	1, 6	4
3	6	2, 5	4
4	5		0
4	6		0
5	6	3, 4	4

$\Rightarrow 24$ such matrices $\Rightarrow 49$ matrices

69.(B)

a	b	c	N
0	0	0	1
1	6		0
2	5		0
3	4		0
4	3		0
5	2		0
6	1		0

70.(B) $\begin{vmatrix} 1-\lambda & -2/3 & 0 \\ -3 & 2-\lambda & 1 \\ 3 & -1 & -\lambda \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2/3 & 0 \\ 0 & 1-\lambda & 1-\lambda \\ 3 & -1 & -\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda) \begin{vmatrix} 1-\lambda & -2/3 & 0 \\ 0 & 1 & 1 \\ 3 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda) \left[(1-\lambda)^2 - 2 \right] = 0 \Rightarrow \lambda = 1, 1 \pm \sqrt{2}$$

71.(B) Matrix A satisfies

$$(I - A)((I - A)^2 - 2I) = 0$$

$$(I - A)(A^2 - 2A - I) = 0$$

$$\Rightarrow A^3 - 3A^2 + A + I = 0 \quad \Rightarrow A^2 - 3A + I + A^{-1} = 0$$

$$\Rightarrow A^{-1} = -(A^2 - 3A + I) = -A^2 + 3A - I = -\frac{1}{3}(A^3 + A + I) + 3A - I = -\frac{1}{3}A^3 + \frac{8}{3}A - \frac{4}{3}I$$

72.(D)
$$\begin{pmatrix} 1 & -\frac{2}{3} & 0 \\ -3 & 2 & 1 \\ 3 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

73.(B) 6 non singular, symmetric matrices

74.(C) For infinitely many solutions, A must be singular matrix

75.(B)
$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow z = -1, x + y = 0 \text{ infinitely many solutions}$$

$$\begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow y + z = 1, x = -1, x = 0 \text{ no solution}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow y = -1, x + z = -1, y = 0 \text{ no solution}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \text{no solution}$$

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \text{no solution}$$

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow x + y = -1 \text{ infinitely many solution.}$$

76.(D) $|Adj(A)| = |A|^2$

77.(A)
$$\begin{vmatrix} A & B & C \\ B & C & A \\ C & A & B \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 \Rightarrow \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \pm 7$$

$$= 3abc - a^3 - b^3 - c^3 = -(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) \leq 0 \text{ as } a+b+c > 0$$

78.(B) $\text{Adj}(\text{Adj } A) = |A|^{n-2} A$

$$\text{Adj}(\text{Adj}(\text{Adj } A)) = |\text{Adj } A|^{n-2} \text{Adj } A = |A|^{(n-1)(n-2)} \text{Adj}(A)$$

79.(C), 80.(D), 81.(B)

In given determinant, applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_2$, we get

$$f(x) = \begin{vmatrix} x+c_1 & a-c_1 & 0 \\ x+b & c_2-b & a-c_2 \\ x+b & 0 & c_3-b \end{vmatrix} = x \begin{vmatrix} 1 & a-c_1 & 0 \\ 1 & c_2-b & a-c_2 \\ 1 & 0 & c_3-b \end{vmatrix} + \begin{vmatrix} c_1 & a-c_1 & 0 \\ b & c_2-b & a-c_2 \\ b & 0 & c_3-b \end{vmatrix}$$

So, $f(x)$ is linear. Let $f(x) = Px + Q$. Then $f(-a) = -aP + Q$, $f(-b) = -bP + Q$

$$\text{Then, } f(0) = 0 \times P + Q \Rightarrow Q = \frac{bf(-a) - af(-b)}{(b-a)} \quad \dots (1)$$

$$\text{Also, } f(-a) = \begin{vmatrix} c_1-a & 0 & 0 \\ b-a & c_2-a & 0 \\ b-a & b-a & c_3-a \end{vmatrix} = (c_1-a)(c_1-a)(c_3-a)$$

Similarly,

$$f(-b) = (c_1-b)(c_2-b)(c_3-b)$$

$$g(x) = (c_1-x)(c_2-x)(c_3-x) \Rightarrow g(a) = f(-a) \text{ and } g(b) = f(-b)$$

$$\text{Now from (1), we get } f(0) = \frac{bg(a) - ag(b)}{(b-a)}$$

82.(C), 83.(A), 84.(D)

The system of equations

$$-x + cy + bz = 0 \quad \dots (1)$$

$$cx - y + az = 0 \quad \dots (2)$$

$$bx + ay - z = 0 \quad \dots (3)$$

$$\text{has a nonzero solution if } \Delta = \begin{vmatrix} -1 & c & b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$$\Rightarrow a^2 + b^2 + c^2 + 2abc - 1 = 0$$

$$\text{or } a^2 + b^2 + c^2 + 2abc = 1 \quad \dots (4)$$

Then clearly the system has infinitely many solutions. From (1) and (2), we have

$$\frac{x}{ac+b} = \frac{y}{bc+a} = \frac{z}{1-c^2}$$

$$\therefore \frac{x^2}{(ac+b)^2} = \frac{y^2}{(bc+a)^2} = \frac{z^2}{(1-c^2)^2}$$

$$\text{or } \frac{x^2}{(1-a^2)(1-c^2)} = \frac{y^2}{(1-b^2)(1-c^2)} = \frac{z^2}{(1-c^2)^2} \quad [\text{from (4)}]$$

$$\text{or } \frac{x^2}{1-a^2} = \frac{y^2}{1-b^2} = \frac{z^2}{1-c^2} \quad \dots (5)$$

From (5), we see that $1-a^2, 1-b^2, 1-c^2$ are all positive or all negative. Given that one of a, b, c is proper fraction, so $1-a^2 > 0, 1-b^2 > 0, 1-c^2 > 0$, which gives

$$a^2 + b^2 + c^2 < 3 \quad \dots (6)$$

Using (4) and (6), we get

$$1 < 3 + 2abc \quad \text{or} \quad abc > -1 \quad \dots (7)$$

85.(A), 86.(B), 87.(C)

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} = 2\lambda + 3 + 2 - 2 - \lambda - 6 = \lambda - 3$$

$$\Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ \mu & 2 & \lambda \end{vmatrix} = 12\lambda + 3\mu + 20 - 2\mu - 10\lambda - 36$$

$$= 2\lambda + \mu - 16$$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 10 & 3 \\ 1 & \mu & \lambda \end{vmatrix} = 10\lambda + 18 + \mu - 10 - 3\mu - 6\lambda$$

$$= 4\lambda - 2\mu + 8$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 1 & 2 & \mu \end{vmatrix} = 2\mu + 10 + 12 - 12 - \mu - 20 = \mu - 10$$

Thus, the system has unique solutions if $\Delta \neq 0$ or $\lambda \neq 3$ and the system has infinite solutions if $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$ or $\lambda = 3$ and $\mu = 10$. System has no solution if $\Delta = 0$ and at least one of $\Delta_1, \Delta_2, \Delta_3$ is nonzero or $\lambda = 3$ and $\mu \neq 10$.

88.(D), 89.(C), 90.(C)

$$\text{Let, } \begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix} = A + Bx + Cx^2 + \dots$$

$$\text{Putting } x = 0, \text{ we get } A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Now differentiating both sides with respect to x and putting $x = 0$, we get

$$B = \begin{vmatrix} a & 2b & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 2b \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2b & 0 & a \end{vmatrix} = 0$$

Hence, coefficient of x is 0. Since $f(x) = 0$ and $f'(0) = 0$, $x = 0$ is a repeating root of the equation $f(x) = 0$.

91.(C), 92.(D), 93.(B)

$$\begin{vmatrix} x & n & r \\ m & y & r \\ m & n & z \end{vmatrix} = 0$$

Applying $R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$, we get $\begin{vmatrix} x-m & n-y & 0 \\ 0 & y-n & r-z \\ m & n & z \end{vmatrix} = 0$

$$\Rightarrow (x-m)(y-n)z + (n-y)(r-z)m - n(r-z)(x-m) = 0$$

Dividing by $(x-m)(y-n)(z-r)$, we have

$$\frac{z}{z-r} + \frac{m}{x-m} + \frac{n}{y-n} = 0 \quad \text{or} \quad \frac{z}{z-r} + \frac{m}{x-m} + \frac{n}{y-n} = 0$$

$$\text{or} \quad \frac{z}{z-r} + \frac{m}{x-m} + 1 + \frac{n}{y-n} + 1 = 2 \quad \text{or} \quad \frac{z}{z-r} + \frac{x}{x-m} + \frac{y}{y-n} = 2 = 2$$

$$\text{or} \quad \frac{z}{z-r} - 1 + \frac{x}{x-m} - 1 + \frac{y}{y-n} - 1 = -1 \quad \text{or} \quad \frac{m}{x-m} + \frac{n}{y-n} + \frac{r}{z-r} = -1$$

Now, $A.M. \geq G.M.$

$$\Rightarrow \frac{\frac{z}{z-r} + \frac{x}{x-m} + \frac{y}{y-n}}{3} \geq \left(\frac{z}{z-r} \frac{x}{x-m} \frac{y}{y-n} \right)^{1/3} \quad \text{or} \quad \frac{z}{z-r} \frac{x}{x-m} \frac{y}{y-n} \leq \frac{8}{27}$$

94.(B), 95.(D), 96.(A)

$$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 - C_3$, $C_2 \rightarrow C_2 - C_3$

$$f'(x) = \begin{vmatrix} -(b+1) & -(b+2) & 2ax+b+1 \\ (b+1) & (b+2) & -1 \\ b & b+1 & 2ax+b \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + R_2$ and $R_3 \rightarrow R_3 - R_2$, we get

$$f'(x) = \begin{vmatrix} 0 & 0 & 2ax+b \\ b+1 & b+2 & -1 \\ -1 & -1 & 2ax+b+1 \end{vmatrix} = (2ax+b)[-b-1+b+2]$$

$$\therefore f'(x) = 2ax + b$$

$$\therefore f(x) = ax^2 + bx + c$$

$$f(0) = 2 \Rightarrow c = 2$$

$$f(1) = 1 \Rightarrow a + b + 2 = 1 \Rightarrow a + b = -1$$

$$f'(5/2) = 0 \Rightarrow 5a + b = 0$$

$$\Rightarrow a = 1/4, b = -5/4$$

$$\text{Hence, } f(x) = \frac{1}{4}x^2 - \frac{5}{4}x + 2$$

Clearly, discriminant (D) of the equation $f(x) = 0$ is less than 0.

Hence, $f(x) = 0$ has imaginary roots. Also, $f(2) = 1/2$. And minimum value of $f(x)$ is

$$\frac{\frac{25}{16} - 4 \cdot \frac{1}{4}(2)}{4 \cdot \frac{1}{4}} = \frac{7}{16}$$

Hence, range of the $f(x)$ is $\left[\frac{7}{16}, \infty\right)$.

97.(ABC)

$$\begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 8 & 2 & 7 \\ 4 & 1 & -2 \\ 4 & 1 & -2 \end{vmatrix} = 0 \quad [R_3 \rightarrow R_3 - R_2 \text{ and } R_2 \rightarrow R_2 - R_1]$$

$$\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ac \\ 1/c & c^2 & ab \end{vmatrix} = \frac{1}{abc} \begin{vmatrix} 1 & a^3 & abc \\ 1 & b^3 & abc \\ 1 & c^3 & abc \end{vmatrix} \quad [R_1 \rightarrow aR_1, R_2 \rightarrow bR_2, R_3 \rightarrow cR_3]$$

$$= \frac{abc}{abc} \begin{vmatrix} 1 & a^3 & 1 \\ 1 & b^3 & 1 \\ 1 & c^3 & 1 \end{vmatrix} = 0 \quad [\text{taking } abc \text{ common from } C_3]$$

$$\begin{vmatrix} a+b & 2a+b & 3a+b \\ 2a+b & 3a+b & 4a+b \\ 4a+b & 5a+b & 6a+b \end{vmatrix} = \begin{vmatrix} a+b & 2a+b & 3a+b \\ a & a & a \\ 2a & 2a & 2a \end{vmatrix} = 0 \quad [R_2 \rightarrow R_3 - R_2, R_2 \rightarrow R_2 - R_1]$$

$$\begin{vmatrix} 2 & 43 & 6 \\ 7 & 35 & 4 \\ 3 & 17 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 6 \\ 7 & 7 & 4 \\ 3 & 3 & 2 \end{vmatrix} \quad [C_2 \rightarrow C_2 - 7C_3]$$

$$= \begin{vmatrix} 1 & 1 & 6 \\ 0 & 7 & 4 \\ 0 & 3 & 2 \end{vmatrix} = 2 \quad [C_1 \rightarrow C_1 - C_2]$$

98.(AC) $g(x) = \begin{vmatrix} a^{-x} & e^{\log_e a^x} & x^2 \\ a^{-3x} & e^{\log_e a^{3x}} & x^4 \\ a^{-5x} & e^{\log_e a^{5x}} & 1 \end{vmatrix} = \begin{vmatrix} a^{-x} & a^x & x^2 \\ a^{-3x} & a^{3x} & x^4 \\ a^{-5x} & a^{5x} & 1 \end{vmatrix} \left(e^{\log a^x} = a^x \right)$

$$\Rightarrow g(-x) = \begin{vmatrix} a^x & a^{-x} & x^2 \\ a^{3x} & a^{-3x} & x^4 \\ a^{5x} & a^{-5x} & 1 \end{vmatrix} = - \begin{vmatrix} a^{-x} & a^x & x^2 \\ a^{-3x} & a^{3x} & x^4 \\ a^{-5x} & a^{5x} & 1 \end{vmatrix} = -g(x) \quad (\text{interchanging 1st and 2nd columns})$$

$$\Rightarrow g(x) + g(-x) = 0$$

$$\Rightarrow g(x) \text{ is an odd function}$$

Hence, the graph is symmetrical about origin. Also, $g_4(x)$ is an odd function [where $g_4(x)$ is fourth derivative of $g(x)$]. Hence,

$$g_4(x) = -g_4(-x) \Rightarrow g_4(0) = -g_4(0) \Rightarrow g_4(0) = 0$$

99.(CD) Applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get

$$\begin{aligned}
 \Delta &= \begin{vmatrix} a+b-x & a & b \\ a+b-x & -x & a \\ a+b-x & b & -x \end{vmatrix} = (a+b-x) \begin{vmatrix} 1 & a & b \\ 1 & -x & a \\ 1 & b & -x \end{vmatrix} \\
 &= (a+b-x) \begin{vmatrix} 1 & a & b \\ 0 & -x-a & a-b \\ 0 & b-a & -x-b \end{vmatrix} \quad [\text{Applying } R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1] \\
 &= (a+b-x)[x+a](x+b) + (a-b)^2 \quad [\text{expanding along } C_1] \\
 &= (a+b-x)[x^2 + (a+b)x + a^2 + b^2 - ab]
 \end{aligned}$$

100.(BD) Applying $C_1 \rightarrow C_1 - (\cot \phi)C_2$, we get

$$\begin{aligned}
 \Delta &= \begin{vmatrix} 0 & \sin \theta \sin \phi & \cos \theta \\ 0 & \cos \theta \sin \phi & -\sin \theta \\ -\sin \theta / \sin \phi & \sin \theta \cos \phi & 0 \end{vmatrix} = -\frac{\sin \theta}{\sin \phi} [-\sin \phi \sin^2 \theta - \cos^2 \theta \sin \phi] \quad [\text{expanding along } C_1] \\
 &= \sin \theta
 \end{aligned}$$

Which is independent of ϕ . Also, $\frac{d\Delta}{d\theta} = \cos \theta \Rightarrow \frac{d\Delta}{d\theta} \Big|_{\theta=\pi/2} = \cos(\pi/2) = 0$

101.(AC) $\begin{vmatrix} \cos \alpha & -\sin \alpha & 1 \\ \sin \alpha & \cos \alpha & 1 \\ \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 1 \end{vmatrix} = \begin{vmatrix} \cos \alpha & -\sin \alpha & 1 \\ \sin \alpha & \cos \alpha & 1 \\ 0 & 0 & 1 + \sin \beta - \cos \beta \end{vmatrix}$ [applying

$$R_3 \rightarrow R_3 - R_1(\cos \beta) + R_2(\sin \beta)]$$

$$= (1 + \sin \beta - \cos \beta)(\cos^2 \alpha + \sin^2 \alpha) = 1 + \sin \beta - \cos \beta \quad \text{which is independent of } \alpha$$

102.(AB) $\Delta = \frac{1}{a} \begin{vmatrix} a^2 + ax & ab & ac \\ a^2b & b^2 + x & bc \\ a^2c & bc & c^2 + x \end{vmatrix}$

Applying $C_1 \rightarrow C_1 + bC_2 + cC_3$ and taking $a^2 + b^2 + c^2 + x$ common, we get

$$\Delta = \frac{1}{a}(a^2 + b^2 + c^2 + x) \begin{vmatrix} a & ab & ac \\ b & b^2 + x & bc \\ c & bc & c^2 + x \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - bC_1$ and $C_3 \rightarrow C_3 - cC_1$, we get

$$\begin{aligned}
 \Delta &= \frac{1}{a}(a^2 + b^2 + c^2 + x) \begin{vmatrix} a & 0 & 0 \\ b & x & 0 \\ c & 0 & x \end{vmatrix} \\
 &= \frac{1}{a}(a^2 + b^2 + c^2 + x)(ax^2) = x^2(a^2 + b^2 + c^2 + x)
 \end{aligned}$$

Thus Δ is divisible by x and x^2

103.(ABC) Applying $R_3 \rightarrow R_3 - xR_2$ and $R_2 \rightarrow R_2 - xR_1$, we get

$$f(x) = \begin{vmatrix} a & -1 & 0 \\ 0 & a+x & -1 \\ 0 & 0 & a+x \end{vmatrix} = a(a+x)^2$$

$$\text{Hence, } f(2x) - f(x) = a[(a+2x)^2 - (a+x)^2] = a(a+2x-a)(a+2x+a+x) = ax(2a+3x)$$

104.(ABC) Operating $C_1 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & ac & bc \\ 1 & ad & bd \\ 1 & ae & be \end{vmatrix} = ab \begin{vmatrix} 1 & c & c \\ 1 & d & d \\ 1 & e & e \end{vmatrix} = ab(0) = 0$$

105.(AB) By partial fractions, we have

$$g(x) = \frac{f(a)}{(x-a)(a-b)(a-c)} + \frac{f(b)}{(b-a)(x-b)(b-c)} + \frac{f(c)}{(c-a)(c-b)(x-c)}$$

$$= \frac{1}{(a-b)(b-c)(c-a)} \times \left[\frac{f(a)(c-b)}{(x-a)} + \frac{f(b)(a-c)}{(x-b)} + \frac{f(c)(b-a)}{(x-c)} \right]$$

$$= \begin{vmatrix} 1 & a & f(a)/(x-a) \\ 1 & b & f(b)/(x-b) \\ 1 & c & f(c)/(x-c) \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\Rightarrow \int g(x)dx = \begin{vmatrix} 1 & a & f(a)\log|x-a| \\ 1 & b & f(b)\log|x-b| \\ 1 & c & f(c)\log|x-c| \end{vmatrix} + k \text{ and } \frac{dg(x)}{dx} = \begin{vmatrix} 1 & a & -f(a)(x-a)^{-2} \\ 1 & b & -f(b)(x-b)^{-2} \\ 1 & c & -f(c)(x-c)^{-2} \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & f(a)(x-a)^{-2} \\ 1 & b & f(b)(x-b)^{-2} \\ 1 & c & f(c)(x-c)^{-2} \end{vmatrix} + \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

106.(BC) $\Delta'(x) = \begin{vmatrix} 2x+4 & 2x+4 & 13 \\ 4x+5 & 4x+5 & 26 \\ 16x-6 & 16x-6 & 104 \end{vmatrix} + \begin{vmatrix} x^2+4x-3 & 2 & 13 \\ 2x^2+5x-9 & 4 & 26 \\ 8x^2-6x+1 & 16 & 104 \end{vmatrix} = 0 + 2 \times 13 \times (0) = 0$

$$\Rightarrow \Delta(x) = \text{constant} \Rightarrow a = 0, b = 0, c = 0$$

107.(AC) $f(\theta) = \sin^3 \theta + \cos^3 \theta - \cos \theta (\sin \theta + \cos \theta)$

$$= (\sin \theta + \cos \theta)^3 - 4 \sin \theta \cos \theta (\sin \theta + \cos \theta)$$

$$= (\sin \theta + \cos \theta)[1 - \sin 2\theta]$$

$$\text{Now, } f(\theta) = 0$$

$$\Rightarrow \tan \theta = -1 \text{ or } \sin 2\theta = 1$$

$$\Rightarrow f(\theta) = 0 \text{ has 2 real solutions in } [0, 2\pi]$$

$$\text{Also, } \frac{f(\theta)}{1 - \sin 2\theta} = \sin \theta + \cos \theta \in [-\sqrt{2}, \sqrt{2}]$$

108.(D) Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin(B+A)\sin(B-A) & \frac{\sin(A-B)}{\sin A \sin B} & 0 \\ \sin(C+A)\sin(C-A) & \frac{\sin(A-C)}{\sin A \sin C} & 0 \end{vmatrix} \quad \left[\because \cot \alpha - \cot \beta = \frac{\sin(\beta - \alpha)}{\sin \alpha \sin \beta} \right]$$

Expanding along C_3 , we get

$$\begin{aligned} \Delta &= \frac{\sin(A-B)\sin(A-C)}{\sin A} \left[-\frac{\sin(B+A)}{\sin C} + \frac{\sin(C+A)}{\sin B} \right] \\ &= \frac{\sin(A-B)\sin(A-C)}{\sin A} \left[-\frac{\sin(\pi-C)}{\sin C} + \frac{\sin(\pi-B)}{\sin B} \right] = \frac{\sin(A-B)\sin(A-C)}{\sin A} \left[-\frac{\sin C}{\sin C} + \frac{\sin B}{\sin B} \right] = 0 \end{aligned}$$

109.(AC)
$$\begin{vmatrix} {}^x C_r & {}^{n-1} C_r & {}^n C_r \\ {}^{x+1} C_r & {}^n C_r & {}^{n+1} C_r \\ {}^{x+2} C_r & {}^{n+1} C_r & {}^{n+2} C_r \end{vmatrix} = 0 \quad \dots\dots\dots(1)$$

$$\Rightarrow \begin{vmatrix} \frac{x!}{r!(x-r)!} & \frac{(n+1)!}{r!(n-r-1)!} & \frac{n!}{r!(n-r)!} \\ \frac{(x+1)!}{r!(x+1-r)!} & \frac{n!}{r!(n-r)!} & \frac{(n+1)!}{r!(n-r+1)!} \\ \frac{(x+2)!}{r!(x+2-r)!} & \frac{(n+1)!}{r!(n+1-r)!} & \frac{(n+2)!}{r!(n-r+2)!} \end{vmatrix} = 0$$

Taking $\frac{x!}{r!(x+2-r)!}$ common from C_1 , we have quadratic equation in x .

Now in (1), if we put $x = n-1$, C_1 and C_2 are the same, hence,

$x = n-1$ is one root of the equation.

If we put $x = n$, then C_1 and C_3 are same. Hence, $x = n$ is the other root.

110.(AB) Applying $C_3 \rightarrow C_3 - xC_2$, $C_2 \rightarrow C_2 - xC_1$, we obtain

$$\Delta(x) = \begin{vmatrix} 3 & 0 & 2a^2 \\ 3x & 2a^2 & 4a^2x \\ 3x^2 + 2a^2 & 4a^2x & 6a^2x^2 + 2a^2 \end{vmatrix}$$

Applying $C_3 \rightarrow C_3 - xC_2$, we get

$$\Delta(x) = 4a^4 \begin{vmatrix} 3 & 0 & 1 \\ 3x & 1 & x \\ 3x^2 + 2a^2 & 2x & x^2 + 2a^2 \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 - 3C_3$, we get

$$\Delta(x) = 4a^4 \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & x \\ -4a^2 & 2x & x^2 + 2a^2 \end{vmatrix} = 16a^6$$

111.(BC)

In the left-hand determinant, each element is the cofactor of the elements of the determinant

$$\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} = \Delta^* (\text{say})$$

Hence,

$$\Delta^{*2} = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} = \begin{vmatrix} x^2 + y^2 + z^2 & xy + yz + zx & xz + yx + zx \\ \Sigma xy & \Sigma x^2 & \Sigma xy \\ \Sigma xy & \Sigma xy & \Sigma x^2 \end{vmatrix}$$

$$= \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix} \quad [\because x^2 + y^2 + z^2 = r^2, xy + yz + zx = u^2]$$

112.(AC)

$$f(n) = \begin{vmatrix} n & n+1 & n+2 \\ n! & (n+1)! & (n+2)! \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} n & 1 & 1 \\ n! & nn! & (n+1)(n+1)! \\ 1 & 0 & 0 \end{vmatrix} \quad [\text{Applying}]$$

$$C_3 \rightarrow C_3 - C_2 \text{ and } C_2 \rightarrow C_2 - C_1]$$

$$= (n+1)(n+1)! - nn! = n![(n+1)^2 - n] = n!(n^2 + n + 1)$$

Thus, $f(n)$ is divisible by $n!$ and $n^2 + n + 1$

113.(ABC)

$$\text{We have } \begin{vmatrix} bc & ca & ab \\ ca & ab & bc \\ ab & bc & ca \end{vmatrix} = 0$$

$$\text{Or } (ab)^3 + (bc)^3 + (ca)^3 - 3(ab)(bc)(ca) = 0$$

$$\text{Or } (ab + bc\omega + ca\omega^2)(ab\omega + bc\omega^2 + ca)(ab\omega^2 + bc\omega + ca) = 0$$

$$\text{Or } ab + bc\omega^2 + ca\omega = 0, ab\omega + bc\omega^2 + ca = 0, ab\omega^2 + bc\omega + ca = 0$$

$$\Rightarrow \frac{1}{c\omega^2} + \frac{1}{a} + \frac{1}{b\omega} = 0, \frac{1}{c\omega} + \frac{1}{a} + \frac{1}{b\omega^2} = 0, \frac{1}{c} + \frac{1}{a\omega} + \frac{1}{b\omega^2} = 0$$

$$\Rightarrow \frac{1}{a} + \frac{1}{b\omega} + \frac{1}{c\omega^2} = 0, \frac{1}{a} + \frac{1}{b\omega^2} + \frac{1}{c\omega} = 0, \frac{1}{a\omega} + \frac{1}{b\omega^2} + \frac{1}{c} = 0$$

114.(AB)

$$\begin{vmatrix} 1 & k & 3 \\ k & 2 & 2 \\ 2 & 3 & 4 \end{vmatrix} = 0$$

$$\text{Or } 8 + 4k + 9k - 12 - 4k^2 - 6 = 0 \quad \text{Or } 4k^2 - 13k + 10 = 0$$

$$\text{Or } 4k^2 - 8k - 5k + 10 = 0 \quad \text{Or } (2k - 5)(k - 2) = 0 \quad \text{Or } k = 5/2, 2$$

115.(AB)

Applying $R_1 \rightarrow R_1 + \sin \phi(R_2) + \cos \phi(R_3)$

$$f(x) = \Delta = \begin{vmatrix} 0 & 0 & \cos 2\phi + 1 \\ \sin \theta & \cos \theta & \sin \phi \\ -\cos \phi & \sin \theta & \cos \phi \end{vmatrix} = (\cos 2\phi + 1)(\sin^2 \theta + \cos^2 \theta) = (1 + \cos 2\phi)$$

Hence, Δ is independent of θ

116.(AC) $\Delta = \begin{vmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = 0$

$$\Delta_x = \begin{vmatrix} a & 1 & 1 \\ b & -2 & 1 \\ c & 1 & -2 \end{vmatrix} = 3a + 3b + 3c$$

$$\Delta_y = \begin{vmatrix} -2 & a & 1 \\ 1 & b & 1 \\ 1 & c & -2 \end{vmatrix} = 3a + 3b + 3c$$

$$\Delta_z = \begin{vmatrix} -2 & 1 & a \\ 1 & -2 & b \\ 1 & 1 & c \end{vmatrix} = 3a + 3b + 3c$$

117.(AB) $\Delta = \begin{vmatrix} a & a^2 & 0 \\ 1 & 2a+b & (a+b) \\ 0 & 1 & 2a+3b \end{vmatrix} = a \begin{vmatrix} 1 & a & 0 \\ 1 & (2a+b) & (a+b) \\ 0 & 1 & 2a+3b \end{vmatrix} = a(a+b) \begin{vmatrix} 1 & a & 0 \\ 0 & 1 & (a+b) \\ 0 & 1 & 2a+3b \end{vmatrix} (R_2 \rightarrow R_2 - R_1)$

$$= a(a+b)(2a+3b-a-b) = a(a+b)(a+2b)$$

118.(C) Symmetric $\begin{bmatrix} a & c \\ c & b \end{bmatrix} \Rightarrow 7 \times 7 \times 1 \times 7 = 7^3$ matrices

Skew symmetric $\begin{bmatrix} 0 & c \\ -c & 0 \end{bmatrix} \Rightarrow 7 \Rightarrow 7^3 + 7 - 1 = 349$ matrices

119.(D) $|C| = |A|^{(3-1)^2} = |A|^4$

120.(D) $a \begin{vmatrix} a & b & c \\ -d & 0 & f \\ -e & -f & 0 \end{vmatrix} - b \begin{vmatrix} a & b & c \\ 0 & d & e \\ -e & -f & 0 \end{vmatrix} + c \begin{vmatrix} a & b & c \\ 0 & d & e \\ -d & 0 & f \end{vmatrix}$

$$= a(af^2 + dfc - ebf) - b(afe - be^2 + cde) + c(adf - bde + cd^2)$$

$$= a^2 f^2 + adfc - aebf - abfe + b^2 e^2 - bcde + acdf - bcde + c^2 d^2 = (be - af - cd)^2$$

121.(CD) $D = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} \times \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} = \alpha\beta\gamma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$

$$\alpha + \beta + \gamma = -2, \quad \alpha\beta + \beta\gamma + \gamma\alpha = 2, \quad \alpha\beta\gamma = -1$$

$$D = \alpha\beta\gamma((\alpha-\beta)(\beta-\gamma)(\gamma-\alpha))^2$$

Roots of given equation are $-1, \omega, \omega^2$

$$D = -1((-1-\omega)(\omega-\omega^2)(\omega^2+1))^2 = -(\omega^2-\omega)^2 = 3$$

122. Clearly $\alpha_1 + \alpha_2 = -b/a$, $\alpha_1\alpha_2 = c/a$, and $\beta_1 + \beta_2 = -q/p$, $\beta_1\beta_2 = r/p$. System of equation $\alpha_1 y + \alpha_2 z = 0, \beta_1 y + \beta_2 z = 0$ has a non-trivial solution. So,

$$\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = 0, \text{ i.e., } \alpha_1\beta_2 - \alpha_2\beta_1 = 0$$

$$\text{Or } \frac{\alpha_1}{\alpha_2} = \frac{\beta_1}{\beta_2} \quad \text{Or } \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} = \frac{\alpha_1 + \alpha_2}{\beta_1 + \beta_2} = \sqrt{\frac{\alpha_1\alpha_2}{\beta_1\beta_2}} \quad \text{Or } \frac{\alpha_1 + \alpha_2}{\beta_1 + \beta_2} = \sqrt{\frac{\alpha_1\alpha_2}{\beta_1\beta_2}}$$

$$\Rightarrow \frac{-b/a}{-q/p} = \sqrt{\frac{c/a}{r/p}} \quad \text{Or } b^2pr = q^2ac$$

$$\begin{aligned} 123. \quad \Delta &= \begin{vmatrix} x & \frac{1}{2}x(x-1) & \frac{1}{6}x(x-1)(x-2) \\ y & \frac{1}{2}y(y-1) & \frac{1}{6}y(y-1)(y-2) \\ z & \frac{1}{2}z(z-1) & \frac{1}{6}z(z-1)(z-2) \end{vmatrix} = \frac{1}{12}xyz \begin{vmatrix} 1 & x-1 & x^2-3x+2 \\ 1 & y-1 & y^2-3y+2 \\ 1 & z-1 & z^2-3z+2 \end{vmatrix} \\ &= \frac{1}{12}xyz \begin{vmatrix} 1 & x & x^2-3x+2 \\ 1 & y & y^2-3y+2 \\ 1 & z & z^2-3z+2 \end{vmatrix} \quad [R_2 \rightarrow R_2 + R_1] \\ &= \frac{1}{12}xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad [R_3 \rightarrow R_3 + 3R_2 - 2R_1] \\ &= \frac{1}{12}xyz(x-y)(y-z)(z-x) \end{aligned}$$

$$124. \quad \text{Let } \Delta = \begin{vmatrix} -2a & a+b & a+c \\ b+a & 2a & c-a \\ c+a & c+b & -2c \end{vmatrix}$$

$$\text{Putting } a+b=0 \text{ or } b=-a, \text{ we get } \Delta = \begin{vmatrix} -2a & 0 & a+c \\ 0 & -2b & b+c \\ c+a & c-a & -2c \end{vmatrix}$$

Expanding along R_1 ,

$$\Delta = -2a \left\{ -4ac - (c-a)^2 - 0 + (a+c) \{ 0 - 2a(c+a) \} \right\} = 2a(c+a)^2 - 2a(c+a)^2 = 0$$

Hence $a+b$ is a factor of Δ . Similarly, $b+c$ and $c+a$ are the factors of Δ .

On expansion of determinant, we can see that each term of the determinant is a homogeneous expression in a, b, c of degree 3 and also R.H.S is a homogeneous expression of degree 3.

$$\therefore \Delta = k(a+b)(b+c)(c+a) = \begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = k(a+b)(b+c)(c+a)$$

$$\text{Putting } a=0, b=1, c=2, \text{ we get } \begin{vmatrix} 0 & 1 & 2 \\ 1 & -2 & 3 \\ 2 & 3 & -4 \end{vmatrix} = k(0+1)(1+2)(2+0)$$

$$\text{Or } 0 - 1(-4 - 6) + 2(3 + 4) = 6k \quad \text{Or } 24 = 6k$$

$$\therefore k = 4$$

$$\text{Hence, } \begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(a+b)(b+c)(c+a)$$

125. Given equation is

$$\begin{bmatrix} x & y \\ z & t \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x^2 + yz & xy + yt \\ zx + tz & zy + t^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow x^2 + yz = 0 \quad (1)$$

$$y(x + t) = 0 \quad (2)$$

$$z(x + t) = 0 \quad (3)$$

$$yz + t^2 = 0 \quad (4)$$

From Eqs. (1) and (4), we have $x^2 = t^2$ or $x = \pm t$

Case I:

If $x = t$, from Eqs. (2) and (3), $y = 0, z = 0$. Then from Eq (1), $x = 0 = t$

Case II:

If $x = -t$, then Eqs. (2) and (3) are satisfied for all values of y and z . if we take $y = -\beta, z = \alpha$ then from Eq. (1).

$$x = \pm \sqrt{\alpha\beta} = -t$$

Obviously, case I is included in case II ($\alpha = 0 = \beta$). Hence, the general solution of the given equation is

$$x = -t = \pm \sqrt{\alpha\beta}, y = -\beta, z = \alpha \Rightarrow \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} \pm \sqrt{\alpha\beta} & -\beta \\ \alpha & \mp \sqrt{\alpha\beta} \end{bmatrix}$$

Where α, β are arbitrary.

126. $A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$

$$\Rightarrow A^2 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix}$$

$$\Rightarrow A^2 + 3A + 2I = \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix} + 3 \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow A^2 + 3A + 2I = O$$

Form Eq. (1), (1)

$$A^3 + 3A^2 + 2A = O$$

$$\text{Or } (A + I)^3 - A = I^3 \quad \text{Or } A = (A + I)^3 - I^3 = (A + I)^3 + (-I)^3$$

$$\Rightarrow B = A + I \text{ and } C = -I$$

$$\therefore B = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \text{ and } C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

127. Let $P = \begin{bmatrix} A & B \\ C & O \end{bmatrix}$

Consider the matrix equation $PX = Q$

Or $\begin{bmatrix} A & B \\ C & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$

Which is equivalent to the following system of equations

$$Ax_1 + Bx_2 = q_1 \quad (1)$$

$$Cx_1 + O = q_2 \quad (2)$$

From Eq. (2) we have

$$Cx_1 = q_2$$

Or $C^{-1}Cx_1 = C^{-1}q_2$

Or $x_1 = q_2 C^{-1} \quad (3)$

Putting the value of x_1 in Eq. (1), we get

$$q_2 AC^{-1} + Bx_2 = q_1$$

Or $q_2 B^{-1}AC^{-1} + B^{-1}Bx_2 = B^{-1}q_1$

Or $x_2 = B^{-1}q_1 - q_2 B^{-1}AC^{-1} \quad (4)$

Both Eqs. (3) and (4) are equivalent to the matrix equation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 0 & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{bmatrix} Q \Rightarrow P^{-1} = \begin{bmatrix} 0 & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{bmatrix}$$

128. Let $P = 1/2(A + A^\theta)$

And $Q = 1/2i(A - A^\theta)$. Then,

$$A = P + iQ \quad (1)$$

Now, $P^\theta = \left\{ \frac{1}{2}(A + A^\theta) \right\}^\theta$

$$= \frac{1}{2}(A^\theta + (A^\theta)^\theta) = \frac{1}{2}(A^\theta + A) = \frac{1}{2}(A + A^\theta) = P$$

Therefore, P is a hermitian matrix. Also

$$Q^\theta = \left\{ \frac{1}{2i}(A - A^\theta) \right\}^\theta$$

$$= \left(\frac{1}{2i} \right) (A - A^\theta)^\theta = -\frac{1}{2i} \{ A^\theta - (A^\theta)^\theta \} = -\frac{1}{2i} (A^\theta - A) = \frac{1}{2i} (A - A^\theta) = Q$$

Therefore, Q is also a Hermitian matrix.

Thus, A can be expressed in the form (1). Since A is unique,

Let $A = R + iS$, where R and S are both Hermitian matrices. We have

$$\begin{aligned} A^\theta &= (R + iS)^\theta = R^\theta + (iS)^\theta \\ &= R^\theta - iS^\theta = R - iS \quad (\text{since } R \text{ and } S \text{ are both Hermitian}) \end{aligned}$$

$$\therefore A + A^\theta = (R + iS) + (R - iS) = 2R$$

$$\Rightarrow R = \frac{1}{2}(A + A^\theta) = P$$

$$\text{Also, } A - A^\theta = (R + iS) - (R - iS) = 2iS \Rightarrow S = \frac{1}{2i}(A - A^\theta) = Q$$

Hence, expression (1) for A is unique.

129. We have

$$A = \begin{bmatrix} 2+3i & 2 & 5 \\ -3-i & 7 & 3-i \\ 3-2i & i & 2+i \end{bmatrix}$$

$$\Rightarrow A^T = \begin{bmatrix} 2+3i & -3-i & 3-2i \\ 2 & 7 & i \\ 5 & 3-i & 2+i \end{bmatrix}$$

$$\Rightarrow A^{\theta} = \begin{bmatrix} 2-3i & -3+i & 3+2i \\ 2 & 7 & -i \\ 5 & 3+i & 2-i \end{bmatrix} \quad \text{Or} \quad A^\theta = \begin{bmatrix} 2-3i & -3+i & 3+2i \\ 2 & 7 & -i \\ 5 & 3+i & 2-i \end{bmatrix}$$

$$\begin{aligned} \therefore A + A^\theta &= \begin{bmatrix} 2+3i & 2 & 5 \\ -3-i & 7 & 3-i \\ 3-2i & i & 2+i \end{bmatrix} + \begin{bmatrix} 2-3i & -3+i & 3+2i \\ 2 & 7 & -i \\ 5 & 3+i & 2-i \end{bmatrix} \\ &= \begin{bmatrix} 4 & -1+i & 8+2i \\ -1-i & 14 & 3-2i \\ 8-2i & 3+2i & 4 \end{bmatrix} \end{aligned} \quad (1)$$

$$\begin{aligned} \text{And } A - A^\theta &= \begin{bmatrix} 2+3i & 2 & 5 \\ -3-i & 7 & 3-i \\ 3-2i & i & 2+i \end{bmatrix} - \begin{bmatrix} 2-3i & -3+i & 3+2i \\ 2 & 7 & -i \\ 5 & 3+i & 2-i \end{bmatrix} \\ &= \begin{bmatrix} 6i & 5-i & 2-2i \\ -5-i & 0 & 3 \\ -2-2i & -3 & 2i \end{bmatrix} \end{aligned} \quad (2)$$

Adding Eqs. (1) and (2), we get

$$\begin{aligned} 2A &= \begin{bmatrix} 4 & -1+i & 8+2i \\ -1-i & 14 & 3-2i \\ 8-2i & 3+2i & 4 \end{bmatrix} + \begin{bmatrix} 6i & 5-i & 2-2i \\ -5-i & 0 & 3 \\ -2-2i & -3 & 2i \end{bmatrix} \\ \text{Hence, } A &= \begin{bmatrix} 2 & -\frac{1}{2} + \frac{i}{2} & 4+i \\ -\frac{1}{2} - \frac{i}{2} & 7 & \frac{3}{2} - i \\ 4-i & \frac{3}{2} + i & 2 \end{bmatrix} + \begin{bmatrix} 3i & \frac{5}{2} - \frac{i}{2} & 1-i \\ -\frac{5}{2} - \frac{i}{2} & 0 & \frac{3}{2} \\ -1-i & -\frac{3}{2} & i \end{bmatrix} \end{aligned}$$

130.(2) System of equations

$$\alpha x + y + z = \alpha - 1 \quad (1)$$

$$x + \alpha y + z = \alpha - 1 \quad (2)$$

$$x + y + \alpha z = \alpha - 1 \quad (3)$$

Since system has no solution.

Therefore, (1) $\Delta = 0$ and (2) $\alpha - 1 \neq 0$

$$\begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = 0, \alpha \neq 1$$

$$R_1 \rightarrow R_1 - R_3, R_2 \rightarrow R_2 - R_3$$

$$\begin{vmatrix} \alpha - 1 & 0 & 1 - \alpha \\ 0 & \alpha - 1 & 1 - \alpha \\ 1 & 1 & \alpha \end{vmatrix} = 0$$

$$\text{or } (\alpha - 1)[\alpha(\alpha - 1) - (1 - \alpha)] + (1 - \alpha)[-(\alpha - 1)] = 0$$

$$\text{or } (\alpha - 1)[\alpha(\alpha - 1) + (\alpha - 1)] + (\alpha - 1)^2 = 0$$

$$\text{or } (\alpha - 1)^2[(\alpha + 1) + 1] = 0 \quad \text{or } \alpha = 1, -2 \text{ or } \alpha = 1, -2$$

Since system has no solution, $\alpha \neq 1$.

$$\therefore \alpha = -2$$

131.(3) $x + y + z = 1$

$$x + 2y + 4z = p$$

$$x + 4y + 10z = p^2$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{vmatrix}$$

$$R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$$

$$= \begin{vmatrix} 0 & -1 & -3 \\ 0 & -2 & -6 \\ 1 & 4 & 10 \end{vmatrix} = 0$$

Since $\Delta = 0$, solution is not unique solution.

The system will have infinite solutions if $\Delta_1 = 0, \Delta_2 = 0, \Delta_3 = 0$

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ p & 2 & 4 \\ p^2 & 4 & 10 \end{vmatrix} = 0$$

$$C_3 \rightarrow C_3 - C_2$$

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 0 \\ p & 2 & 2 \\ p^2 & 4 & 6 \end{vmatrix} = 0$$

$$\text{or } 1(12 - 8) - 1(6p - 2p^2) = 0 \quad \text{or } 4 - 6p + 2p^2 = 0 \quad \text{or } 2(p^2 - 3p + 2) = 0$$

$$\text{or } p^2 - 3p + 2 = 0 \quad \Rightarrow \quad p = 1 \text{ or } 2$$

Also for these values of $p, \Delta_2, \Delta_3 = 0$

$$132.(0) \begin{vmatrix} 3u^2 & 2u^3 & 1 \\ 3v^2 & 2v^3 & 1 \\ 3w^2 & 2w^3 & 1 \end{vmatrix} = 0$$

$$R_1 \rightarrow R_1 - R_2 \text{ and } R_2 \rightarrow R_2 - R_3$$

$$\text{or } \begin{vmatrix} u^2 - v^2 & u^3 - v^3 & 0 \\ v^2 - w^2 & v^3 - w^3 & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} u + v & u^2 + v^2 + uv & 0 \\ v + w & v^2 + w^2 + vw & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$$

$$R_1 \rightarrow R_1 - R_2$$

$$\text{or } \begin{vmatrix} u - w & (u^2 - w^2) + v(u - w) & 0 \\ v + w & v^2 + w^2 + vw & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 1 & u + w + v & 0 \\ v + w & v^2 + w^2 + vw & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$$

$$\text{or } (v^2 + w^2 + vw) - (v + w)[(v + w) + u] = 0 \quad \text{or} \quad v^2 + w^2 + vw - (v + w)^2 - u(v + w) = 0$$

$$\text{or } uv + vw + wu = 0$$

133.(2) Using $C_3 \rightarrow C_3 - (C_1 + C_2)$ in D_1 and D_2 , we have

$$\frac{D_1}{D_2} = \frac{-2b(ad - bc)}{b(ad - bc)} = -2$$

$$134.(5) \Delta = \begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 1 & \sin\theta & 1 \end{vmatrix}$$

$$\text{Applying } R_3 \rightarrow R_3 - R_1$$

$$= \begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 0 & \sin\theta - 3\cos\theta & 0 \end{vmatrix} = -(\sin\theta - 3\cos\theta)(3\cos\theta - \sin\theta) = (3\cos\theta - \sin\theta)^2$$

$$\text{Now, } -\sqrt{9+1} \leq 3\cos\theta - \sin\theta \leq \sqrt{9+1}$$

$$\Rightarrow (3\cos\theta - \sin\theta)^2 \leq 10.1 \Rightarrow \Delta_{\max} = 10$$

$$135.(4) \Delta = x \begin{vmatrix} 1 & x+y & x+y+z \\ 2 & 3x+2y & 4x+3y+2z \\ 3 & 6x+3y & 10x+6y+3z \end{vmatrix}$$

$$= x^2 \begin{vmatrix} 1 & 1 & x+y \\ 2 & 3 & 4x+3y \\ 3 & 6 & 10x+6y \end{vmatrix} \begin{matrix} C_3 \rightarrow C_3 - zC_1 \\ C_2 \rightarrow C_2 - yC_1 \end{matrix} = x^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} [C_3 \rightarrow C_3 - yC_2]$$

$$= x^3(6 - 8 + 3) = 64$$

$$x^3 = 64 \text{ or } x = 4$$

$$136.(1) \Delta_1 = \begin{vmatrix} a_1^2 + 4a_1d & a_1 & d \\ a_2^2 + 4a_2d & a_2 & d \\ a_3^2 + 4a_3d & a_3 & d \end{vmatrix}, [C_3 \rightarrow C_3 - C_2]$$

Where d is the common difference of A.P.

$$= d \begin{vmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{vmatrix} + 4d \begin{vmatrix} a_1 & a_1 & d \\ a_2 & a_2 & d \\ a_3 & a_3 & d \end{vmatrix} = d(a_1 - a_2)(a_2 - a_3)(a_3 - a_1) = -2d^4$$

Similarly, $\Delta_2 = -2d^4$

$$137.(4) \Delta = (xyz)^n \begin{vmatrix} 1 & x^2 & xy^4 \\ 1 & y^2 & y^4 \\ 1 & z^2 & z^4 \end{vmatrix} = (xyz)^n (x^2 - y^2)(y^2 - z^2)(z^2 - x^2)$$

$$\text{Clearly when } n = -4, \Delta = \left(\frac{1}{y^2} - \frac{1}{x^2}\right) \left(\frac{1}{z^2} - \frac{1}{y^2}\right) \left(\frac{1}{x^2} - \frac{1}{z^2}\right)$$

$$138.(2) B = 2.2 \begin{vmatrix} f & d & e \\ n & l & m \\ c & a & b \end{vmatrix}$$

[Taking 2 common from R_2 and C_2]

$$= 2 \begin{vmatrix} 2f & d & e \\ 2n & l & m \\ 2c & a & b \end{vmatrix} = 2 \begin{vmatrix} 2c & a & b \\ 2f & d & e \\ 2n & l & m \end{vmatrix} \quad [R_3 \leftrightarrow R_2, \text{ then } R_2 \leftrightarrow R_1]$$

$$= 2 \begin{vmatrix} a & b & 2c \\ d & e & 2f \\ l & m & 2n \end{vmatrix} = 2A \quad [C_1 \leftrightarrow C_2 \text{ and then } C_2 \leftrightarrow C_3]$$

139.(8) Putting $x = 0, a_0 = 1$

$$(1 + ax + bx^2)^4 = (1 + ax + bx^2)(1 + ax + bx^2)(1 + ax + bx^2)(1 + ax + bx^2)$$

Clearly $a_0 = 1, a_1 = \text{coefficient of } x = a + a + a + a = 4a$

$$a_2 = \text{coefficient of } x^2 = 4b + 6a^2$$

$$\text{Now } \Delta = -(a_0^3 + a_1^3 + a_2^3 - 3a_0a_1a_2)$$

$$\therefore a_0 + a_1 + a_2 \neq 0 \quad \therefore a_0 = a_1 = a_2$$

$$\therefore a_0 = a_1 = a_2$$

$$1 = 4a = 6a^2 + 4b \Rightarrow a = \frac{1}{4}, b = \frac{5}{32}$$

$$140.(1) A^2 - 2A + I_2 = 0$$

$$A^2 = 2A - I$$

$$A^3 = 2A^2 - A = 4A - 2I - A = 3A - 2I$$

$$A^4 = 3A^2 - 2A = 6A - 3I - 2A = 4A - 3I$$

$$A^n = nA - (n-1)I ; \quad A^{2016} = 2016A - 2015I$$

$$141.(8) \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

$$7\sin(3\theta) + 14\cos(2\theta) - 14 = 0$$

$$\sin 3\theta + 2\cos 2\theta - 2 = 0$$

$$\sin 3\theta = 2(2\sin^2 \theta)$$

$$3\sin \theta - 4\sin^3 \theta = 4\sin^2 \theta$$

$$\sin \theta = 0, 4\sin^2 \theta + 4\sin \theta - 3 = 0$$

$$\sin \theta = 0, \frac{1}{2}$$

$$142.(9) A = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ 2 & 2 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix} = -\frac{1}{144}$$

$$143.(8) f(x) = (1 - x + x^2)e^x$$

$$\begin{vmatrix} f(0) & f'(0) & f''(0) \\ f'(0) & f''(0) & f'''(0) \\ f''(0) & f'''(0) & f^{iv}(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 4 & 9 \end{vmatrix} = -8$$

$$144.(6) A^2 = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & 4 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & 4 \\ -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -2 & -6 & -8 \\ -4 & 4 & 4 \\ 0 & 2 & -2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} -2 & -6 & -8 \\ -4 & 4 & 4 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & 4 \\ -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 6 & -8 & -22 \\ -8 & 16 & 0 \\ 2 & 4 & 10 \end{pmatrix}$$

$$\text{Trace } (A^3) = 32 = 2^5 \Rightarrow \text{Number of factors} = 6$$

$$145.(0) |A - I| = |A - AA'| = |A| |I - A'| = |(I - A)^1| = |I - A| = -|A - I|$$

$$146.(2) \begin{vmatrix} \alpha - a & b - \beta & 0 \\ 0 & \beta - b & c - \gamma \\ a & b & \gamma \end{vmatrix} = a(b - \beta)(c - \gamma) - b(\alpha - a)(c - \gamma) + \gamma(\alpha - a)(\beta - b) = 0 \Rightarrow \frac{a}{\alpha - a} + \frac{b}{\beta - b} + \frac{\gamma}{\gamma - c} = 0$$